Strange Loops and Substitution Tilings

Ben Sinclair
Supervisor: John Hunton

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1 Introduction

This research brings together two distinct mathematical topics of recent interest to the mathematical community, *Aperiodic Substitution Tilings* and the underlying mathematics in the artwork of M.C. Escher, with particular focus on the lithograph *Print Gallery* (Figure 1) and Escher's Grid (Figure 12) (which is the grid that *Print Gallery* was imposed on).

A major theme of M.C. Escher's artwork was to utilise a concept called a *strange loop*. Consider the lithograph *Ascending and Descending* given in Figure 1. If one picks any point on the upper set of stairs then travels up (or down) the stairs, one would find themselves back at the point where they started. This illustrates the concept of a strange loop and demonstrates some sort of 'jump' in the hierarchical system of these stairs.

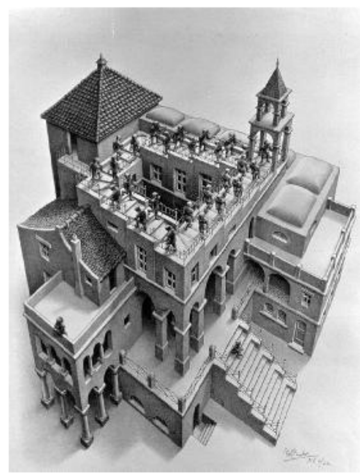

Figure 1: *Print Gallery* (Left) and *Ascending and Descending* (Right) by M.C. Escher

The main objective of this research is to produce a method to construct a substitution tiling that contains different hierarchy levels, so that a strange loop between the hierarchy levels is produced in Escher's Grid (or some variation of Escher's Grid). This will be achieved by analysing and applying the mathematics underlying *Print Gallery*, by providing a formal and generalised definition of a strange loop and by defining a hierarchical system on a substitution map and its corresponding substitution tiling.

In addition, the researcher will then use this method to produce a visual of a distorted tiling that contains strange loops between the hierarchy levels of the distorted tiling.

1.1 A Non-Mathematical Summary of this Research

Note: Anyone who is familiar with reading mathematical reports, please skip this subsection and read Section 2 *and onwards.*

When Escher was creating the lithograph Print Gallery (Figure 1), he wanted to take a standard undistorted square grid invariant under scalar multiplication by 256 and transform this grid to a distorted grid (namely Escher's Grid in Figure 12) with the following properties:

- The original undistorted squares which make up the undistorted grid "[retained] their square appearance" ([1] pg 447).
- "As one goes clockwise around the center, the grid folds onto itself, but expanded by a factor of ... 256" ([1] pg 447).

So what does a square grid invariant under scalar multiplication by a certain factor look like? To visualise this, it is useful to understand a concept called the *Droste Effect*. The Droste Effect occurs when a picture recursively appears within itself. For example, consider the artwork *Going Down Droste* in Figure 2. If we zoom into the centre of this image, we notice that the zoomed in version contains a copy of the original image, and repeating this process will gives another copy of the original image and so on. Therefore this image illustrates the Droste Effect and how much we have to zoom in to get a copy of the original image gives the certain factor as described above. We would then say that this image is imposed on a square grid invariant under scalar multiplication by this factor.

Figure 2: Going Down Droste [8] (Horizontally Flipped)

The analysis given in [1] shows that a mathematical transformation can be constructed from the undistorted grid to the distorted grid (Escher's Grid) which satisfy the properties given on the previous page. In addition, it is shown in [1] that the second property is equivalent to taking all anticlockwise paths in the undistorted grid (where the end point is 256 times further away than the start point) and transforming these paths to single anticlockwise loops in Escher's Grid, where the start and end points are the same point. Figure 3 illustrates this.

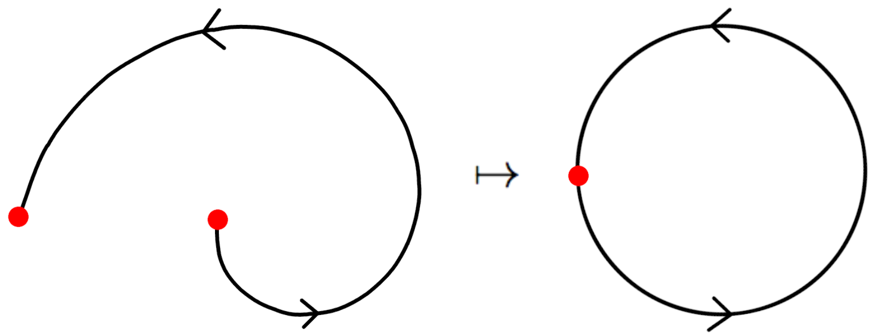

Figure 3: Path (labelled with direction) in the undistorted grid (Left) and how this path is mapped onto Escher's Grid (Right). Red points indicate start and end points of the paths

For instance, consider the orange anticlockwise path travelling up the stairs in Figure 4. If this path is transformed in the same way given in Figure 3, then this path would become a single anticlockwise loop on (some variation of) Escher's Grid (the red points would be 'joined' up) and the entire image would be distorted in a way to reflect this transformation.

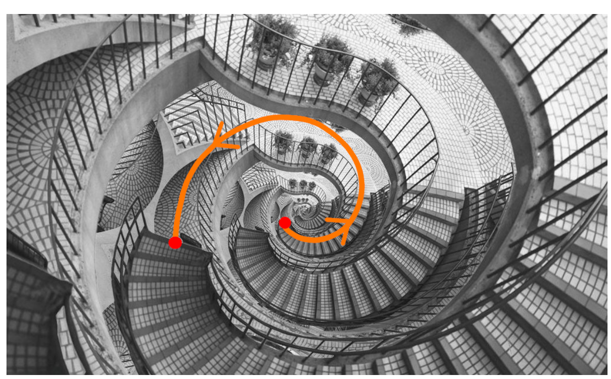

Figure 4: Going Down Droste with orange anticlockwise path (start and end points indicated by red circles)

This provides a motivation for producing strange loops on Escher's Grid. Strange loops were informally defined by Douglas R. Hofstadter as loops such that when "moving upwards (or downwards) through the levels of some hierarchical system, we unexpectedly find ourselves right back where we started" ([5] pg 10). This was illustrated in Section 1 for the lithograph *Ascending and Descending* (Figure 1).

So if the path in the undistorted grid of Figure 3 travelled upwards through the levels of some hierarchy system then returned to the same hierarchy level as the starting point of this path, and since this path is a loop (start and end points are the same point) in Escher's Grid, then this loop would be a strange loop in Escher's Grid.

Figure 4 gives a good illustration of this. As one travels around the orange path, one would ascend a set of stairs which can be thought of as levels of a hierarchy. As discussed previously, this path's start and end points would be joined up in (some variation of) Escher's Grid. Therefore this new loop would represent ascending a set of stairs and then finding ourselves back at the bottom of the stairs. This displays that this new loop is a strange loop.

The researcher is interested in producing a substitution tiling that contain multiple hierarchy levels so that strange loops are produced on Escher's Grid (or variations of Escher's Grid). So we need to be familiar with the following terminology.

A tiling can be described as a 2D image, occupied by tiles, such that there are no gaps on the image and there are no overlaps between the individual tiles (like a jigsaw). An example of this is given in Figure 5.

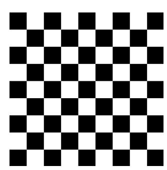

Figure 5: Checkerboard Tiling which uses black and white tiles

Substitution tilings are tilings produced by the 'inflate and subdivide rule'. Consider the yellow tile in Figure 6. When the 'inflate and subdivide rule' is applied to the yellow tile, the tile is scaled up and then this scaled up tile is filled in with the other coloured tiles in Figure 6. Substitution tilings are then produced by repeating this rule indefinitely.

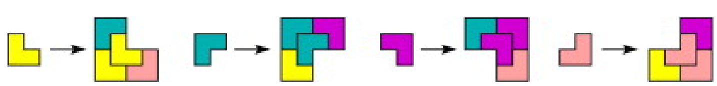

Figure 6: Inflate and subdivide rules for each coloured tile ([4] pg 301)

A hierarchy level of a tile can be described as the number of times the inflate and subdivide rule is applied to a tile. For instance, the set of tiles in Figure 7 which corresponds to the inflate and subdivide rule applied *three* times to the yellow tile of Figure 6 will have a hierarchy level of *three*.

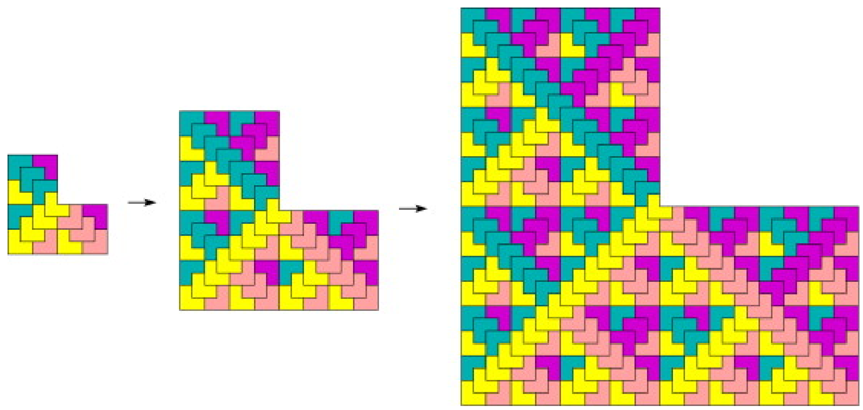

Figure 7: Inflate and subdivide rule applied to the yellow tile of Figure 6 two times (Left), three times (Middle) and four times (Right) ([4] pg 302)

We can now produce a substitution tiling that contains multiple hierarchy levels. Consider the square chair tiling inflate and subdivide rules given in Figure 8. A substitution tiling can be produced by repeating applying the inflate and subdivide rule to the red tile and multiple hierarchy levels can be placed on this substitution tiling by replacing certain tiles with their corresponding inflated and subdivided tiles (but without the inflation).

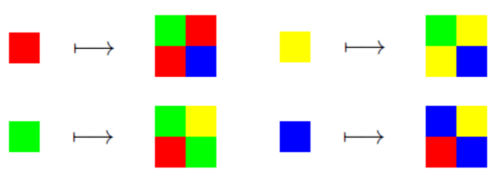

Figure 8: The square chair tiling inflate and subdivide rules for each coloured tile

A square chair substitution tiling generated by applying the inflate and subdivide to the red tile five times and its corresponding tiling which contains multiple hierarchy levels are given in Figure 9.

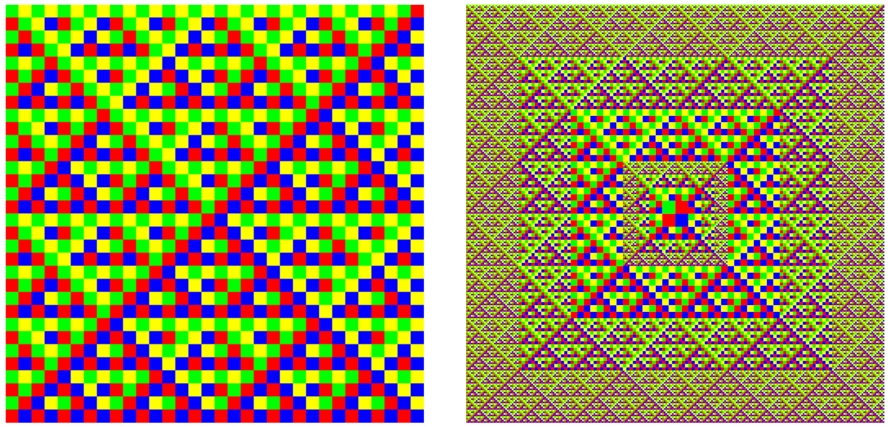

Figure 9: Square Chair substitution tiling (Left) and its corresponding tiling containing multiple hierarchy levels (Right)

To understand what is being represented in the rightmost image of Figure 9, consider Figure 10 which displays the bottom left corners of both images. The red tile in the leftmost image of Figure 10 which is highlighted by a black border is replaced with a set of tiles which correspond to the red tile under the inflate and subdivide rule three times (without the inflation). The set of tiles inside the black border in Figure 10's rightmost image is the replaced set of tiles and since the inflate and subdivide rule was applied three times to obtain this set of tiles, we say that the hierarchy level is three on this set of tiles.

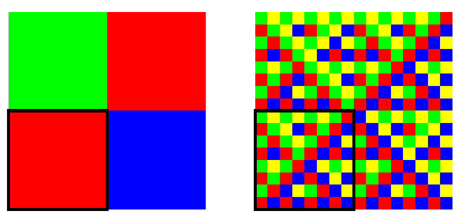

Figure 10: Bottom left corner of the substitution tiling (Left) and its corresponding tiling containing multiple hierarchy levels (Right) of Figure 9 with red tile and its corresponding subdivided tile highlighted by a black border

How each tile is replaced in a substitution tiling is randomly chosen. For example, one could of replaced the highlighted red tile in Figure 10 with the red tile under the inflate and subdivide rule one hundred times (without the inflation) instead of three times. However, the rightmost image of Figure 9 was constructed so that the path on the undistorted grid given in Figure 3 is a strange loop in (a variation of) Escher's Grid. Transforming Figure 9 to (a variation of) Escher's Grid gives a new visual *Escher's Square Chair Tiling* which is given in Figure 11.

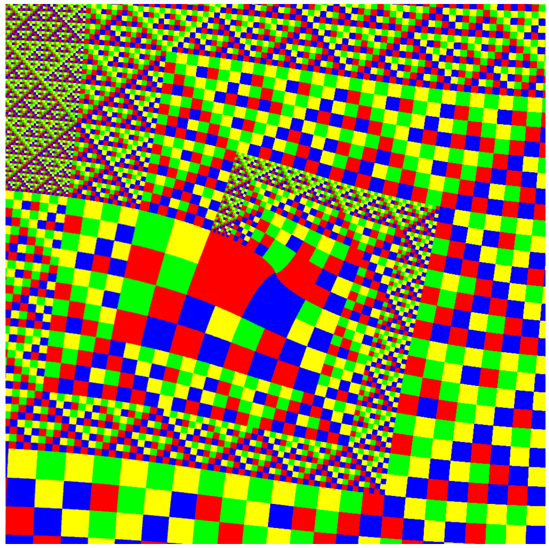

Figure 11: Escher's Square Chair Tiling

Escher's Square Chair Tiling is imposed on a variation of Escher's Grid rather than Escher's Grid since all anticlockwise paths in the undistorted grid where the end point is 4 times further away than the start point (instead of 256) are mapped to single anticlockwise loops in the grid Escher's Square Chair Tiling is imposed on. This implies that as one goes clockwise around the center of Escher's Square Chair Tiling, the image folds onto itself, but expanded by a factor of 4, which reflects the idea that Escher had when he was producing *Print Gallery*.

So the researcher has produced a method to construct substitution tilings that contains different hierarchy levels so that strange loops are produced in Escher's Grid (or some variation of Escher's Grid) and the researcher has used this method to produce a new visual Escher's Square Chair Tiling.

2 The Complex Geometry of Escher's Grid

A large proportion of time during this research project was devoted to a literature review of the article *The Mathematical Structure of Escher's Print Gallery* by B.de Smit and H.W. Lenstra Jr. [1].

To summarise the content of [1], a conformal bijection h was constructed which mapped a distorted grid (namely Escher's Grid see Figure 12) that is invariant under multiplication by $\gamma \in \mathbb{C}^*$, to an undistorted grid (a standard square grid) that is invariant under scalar multiplication by 256. The bijection h represents how the print *Print Gallery* (Figure 1) is mapped to the undistorted grid and mathematically speaking, h is a map from $\mathbb{C}^*/<\gamma>$ to $\mathbb{C}^*/<256>$. The fundamental groups of $\mathbb{C}^*/<\gamma>$ and $\mathbb{C}^*/<256>$ are L_γ and L_{256} respectively where L_δ is defined as $L_\delta := \mathbb{Z}2\pi i + \mathbb{Z}log\delta$. The article then concludes:

- $h : \mathbb{C}^*/<\gamma> \to \mathbb{C}^*/<256>$ where $h(w) = w^\alpha$
- $\gamma = e^{\alpha^{-1} log256}$
- $\alpha = (2\pi i + log256)/2\pi i$ where α satisfies $\alpha L_\gamma = L_{256}$

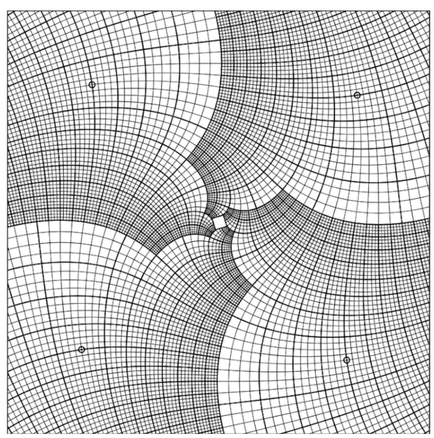

Figure 12: Escher's Grid [1]

Remark 1. The value $\alpha = (2\pi i + log256)/2\pi i$ is derived from the fact that when Escher was constructing his distorted grid, he wanted to take an anticlockwise loop around the origin in the distorted grid and map this to an anticlockwise path around the origin where the start and end point of this path are similar by scale factor 256. This represents the element $2\pi i \in L_\gamma$ being mapped to the element $2\pi i + log256 \in L_{256}$ and using the fact that $\alpha L_\gamma = L_{256}$ implies that $\alpha = (2\pi i + log256)/2\pi i$.

2.1 Producing Variations of Escher's Grid

In the article [1], the value 256 was chosen to represent a decision made by Escher in constructing his distorted grid. Namely, "as one goes clockwise around the center, the grid folds onto itself, but expanded by a factor of $4^4 = 256$" ([1] pg 447). However from a mathematical perspective, 256 is arbitrary. So one can ask if we replace 256 by some $x \in \mathbb{R}_{>0} \setminus \{1\}$, what happens to the bijection between the distorted grid (a variation of Escher's Grid) and the undistorted grid invariant under multiplication by x? So now instead of considering the conformal bijection h as given above, one should consider a conformal bijection:

$$h_x : \mathbb{C}^*/<\gamma_x> \to \mathbb{C}^*/<x>$$

Applying the same methodology as given in [1] pg 449-450 concludes that:

- $h_x(w) = w^{\alpha_x}$
- $\gamma_x = e^{\alpha_x^{-1} logx}$
- $\alpha_x = (2\pi i + logx)/2\pi i$ where α_x satisfies: $\alpha_x L_{\gamma_x} = L_x$

Remark 2. The value $x = 1$ is not considered since the map h_1 is equivalent to the identity map.

Remark 3. The methodology in calculating h_x replicated a part of Escher's procedure as given in Remark 1, namely that the value α_x reflects the idea Escher wanted to map an anticlockwise loop in the distorted grid to an anticlockwise path where the start and end point are similar by scale factor x in the undistorted grid. So mathematically speaking, one can choose where the anticlockwise loop of L_{γ_x} in the distorted grid is mapped to in the undistorted grid's fundamental group L_x; however this has already been examined for $x = 256$, by the authors of [1] and the corresponding grids can be found on the website [2].

To summarise, the bijection h_x is a function that maps a distorted grid (a variation of Escher's Grid) that is invariant under multiplication by γ_x to an undistorted grid that is invaraint under multiplication by x. In addition, h_x maps single anticlockwise loops around the origin to single anticlockwise paths around the origin where the start and end points are similar to each other by a scale factor of x. However, the researcher is interested in mapping undistorted images (in particular undistorted tilings) to distorted images. This implies that the map h_x^{-1} (h_x^{-1} exists since h_x is a bijection) must be used to map these undistorted images to distorted ones. One can show that:

$$h_x^{-1} : \mathbb{C}^*/<x> \to \mathbb{C}^*/<\gamma_x> \text{ where } h_x^{-1}(w) = w^{\alpha_x^{-1}} := e^{\alpha_x^{-1} logw}$$

A proof of this is given in Appendix A.

3 Substitution Tilings

This section is dedicated to providing definitions related to substitution tilings, so that the reader is familiar with the terminology used in the following sections. Note that only tilings that cover $\mathbb{R}^2$ will be considered in this report.

Definition 1. *A tiling T is a subdivision of the plane $\mathbb{R}^2$ into pieces called tiles. Specifically, we have a set of tiles t_i that intersect only on their boundaries, and whose union is the entire plane* ([3] pg 1)*.*

Definition 2. *A prototile set P is a finite set of inequivalent tiles* ([4] pg 299)*.*

For example, one can tile the plane $\mathbb{R}^2 \cong \mathbb{C}$ by using black and white unit squares as prototiles to give a checkerboard tiling (Figure 13). Figure 13 is a tiling since the black and white tiles only intersect on their boundaries.

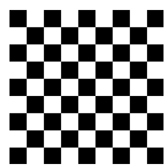

Figure 13: Checkerboard Tiling

Definition 3. *Let $t \in P$ where P is a set of prototiles. A linear transformation σ (that is not the identity map) is a substitution map if $\sigma(t)$ is a union of tiles belonging to the prototile set P. If $\sigma(t)$ is similar to t (same geometric shape but a different size), then σ can be represented as $\sigma(z) = \lambda z$ where λ is defined as the expansion factor of the prototiles and any tiling T generated by this substitution map σ is called a self-similar substitution tiling.*

One can conclude that $|\lambda| > 1$ since if $|\lambda| < 1$, this would show that $\sigma(t)$ has a smaller area than t. This implies that $\sigma(t)$ cannot be the union of prototiles and hence σ is not a substitution map. ($|\lambda| \neq 1$ since σ is not the identity map).

Substitution maps are commonly refered to as the 'inflate and subdivide rule' since the process involves taking a tile, inflating this tile by the expansion factor λ, and then subdividing this inflated tile into its corresponding prototiles. For instance, Figure 14 gives a substitution map σ (each prototile has the same dimensions). This substitution map has an expansion factor of $\lambda = 2$ since for all $t \in P$, $\sigma(t)$ is inflated by a scale factor of 2.

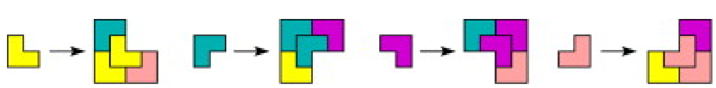

Figure 14: An example of a substitution map ([4] pg 301)

Definition 4. *A patch of a tiling is a finite subset of tiles in a tiling* ([3] pg 1)

Definition 5. *A patch of a tiling corresponding to $\sigma^n(t_i)$ (for some $t_i \in P$ and for some substitution map σ) is called a* <u>*supertile of level n and type i*</u> ([3] pg 11).

For example, Figure 15 shows the supertiles level 2, 3, 4 of the yellow prototile with respect to the substitution map given in Figure 14.

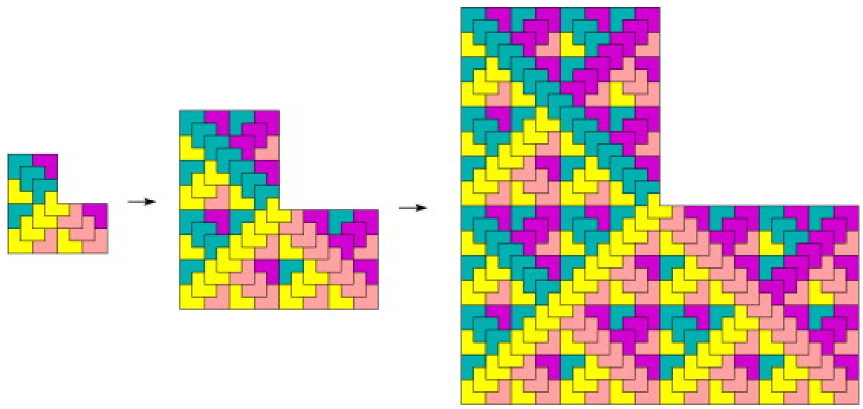

Figure 15: Supertile of the yellow prototile (of Figure 14) level 2 (Left), level 3 (Middle) and level 4 (Right) ([4] pg 302)

In Section 5, the researcher will use self-similar tilings T that are fixed points of some similar substitution map σ, that is $\sigma(T) = T$. To construct these tilings using σ such that the above condition is satisfied, an origin must be defined on the plane $\mathbb{R}^2$ such that a set of prototiles P joined together and $\sigma(P)$ agree on the interior and boundary of P and the boundary of $\sigma(P)$ is further away from the origin than the boundary of P in all directions. By repeating this, we have that $\sigma^n(P)$ and $\sigma(\sigma^n(P))$ agree on the interior and boundary of $\sigma^n(P)$ and the boundary of $\sigma(\sigma^n(P))$ is further away from the origin than the boundary of $\sigma^n(P)$ in all directions. Now the tiling T can be defined as:

$$T = \lim_{n\to\infty} \sigma^n(P)$$

and it immediately follows that $\sigma(T) = T$.

4 Strange Loops

To discuss how certain substitution tilings can be represented on variations of Escher's Grid, one can discuss the phenomenon of a *strange loop*. A informal definition of a strange loop was given by Douglas R. Hofstadter:

Definition (Hofstadter's Definition)**.** *A strange loop can be described as when "moving upwards (or downwards) through the levels of some hierarchical system, we unexpectedly find ourselves right back where we started"* ([5] pg 10).

A strange loop exists in Print Gallery (Figure 1). "What we see is a picture gallery where a young man is standing, looking at a picture of a ship in the harbor of a small town, ... upon one of [the buildings] ... sits a boy, ... while two floors below him is a woman ... [who] sits directly above a picture gallery where a young man is standing, looking at a picture of a ship in the harbor of a small town" ([5] pg 715). The start and end points are on the same level, despite travelling through increasing levels of the hierarchy.

Remark 4. A hierarchy of a substitution map can be defined based on the level of a supertile. For example, a supertile of level $n+1$ has a higher hierarchy level than a supertile of level n.

Since a strange loop is produced in Figure 1, the underlying complex geometry of Escher's Grid (Figure 12) has the potential to allow an undistorted image which contains multiple hierarchy levels to be mapped to and produce a strange loop on Escher's Grid. So one can ask, is there a way to divide a substitution tiling into different hierarchy levels such that a strange loop between these hierarchy levels is produced in Escher's Grid (or some variation of Escher's Grid)?

In order to achieve this, this section will give the strange loop a more formal and generalised definition which will use Hofstadter's definition as a special case of strange loops. Since the researcher is interested in strange loops between hierarchy levels of tiling substitution maps, the strange loop will be defined for *discrete* hierarchy levels instead of continuous hierarchy levels.

Definition 6. *A* <u>*(discrete) hierarchy function*</u> *is a function $f : \mathbb{C} \to \mathbb{N}_0$ which records the hierarchy level of a given point on* $\mathbb{C}$

In Hofstadter's definition, a strange loop is described in terms of levels of a hierarchical system. So the purpose of a discrete hierarchy function is to assign a discrete numbered hierarchy level to each point on $\mathbb{C}$, where level m is 'higher up' on the hierarchy than level n if and only if m is greater than n.

To illustrate this, consider a 2D blueprint of a skyscraper. A discrete hierarchy function can be defined on this image by calling all points on the ground floor level 0, calling all points on the first floor level 1 and so on.

Another discrete hierarchy function is given in Figure 16, where all points in the white region are level 1, all points in the light grey region are level 2 and all points in the dark grey region are level 3.

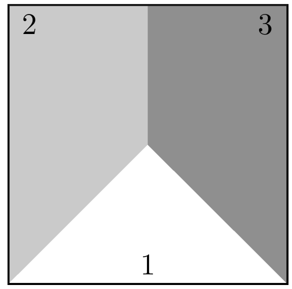

Figure 16: A discrete hierarchy function with levels of each section labelled

For the remainder of this section, discrete hierarchy functions will be called hierarchy functions.

Definition 7. *Let $\alpha : [0,1] \to \mathbb{C}$ be a path, let $f : \mathbb{C} \to \mathbb{N}_0$ be a hierarchy function and let $t \in [0,1]$, then $\alpha(t)$ is a* <u>*strange point (on f)*</u> *if:*

$$\forall\, \delta > 0 \text{ arbitrarily small, } |f \circ \alpha(t+\delta) - f \circ \alpha(t-\delta)| > 1$$

Informally, a strange point (on f) is a point on a path where there is a 'jump' or 'fall' in hierarchy levels of 2 or more with respect to the hierarchy function f. Why classify these points as strange? For the same reason it would be 'strange' to travel from the ground floor of a skyscraper (level 0) to the second floor (level 2) without passing through the first floor (level 1). In this context, a 'jump' in hierarchy levels of 0 (remaining on the same floor) or 1 (moving up or down one floor) is how one traverses a skyscraper. For this reason, any point on a path between two points on the same floor or between two points on different floors is not a strange point.

A visualisation of strange points has been provided in Figure 17 using the hierarchy function given in Figure 16. The blue loop has no strange points since the only points where there is a change in hierarchy levels are from level 2 to 3 and from level 3 to 2 (a change of ± 1). However, the red loop has two points (indicated by black circles) where there is a change in hierarchy levels of ± 2. So these two points are strange points.

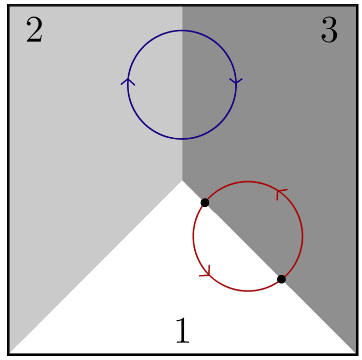

Figure 17: Loops with strange points indicated by solid black circles

For the remaining definitions in this section, $\alpha : [0,1] \to \mathbb{C}$ is an arbitrary path *that does not start and end at a strange point*, and $f : \mathbb{C} \to \mathbb{N}_0$ is an arbitrary hierarchy function.

Definition 8. *Let $\alpha(t)$ be a strange point, then $\alpha(t)$ is an increasing strange point (on f) if $\forall\, \delta > 0$ arbitrarily small, $f \circ \alpha(t+\delta) > f \circ \alpha(t-\delta)$.*

Similarly, $\alpha(t)$ is a decreasing strange point (on f) if $\forall\, \delta > 0$ arbitrarily small, $f \circ \alpha(t+\delta) < f \circ \alpha(t-\delta)$.

An example of an increasing strange point and decreasing strange point is given in Figure 17. The red loop has two strange points. The strange point closest to the origin is a decreasing strange point since as one passes through this point travelling on the loop, the hierarchy level changes from 3 to 1. Similarly, the strange point furthest away from the origin is an increasing strange point since as one passes through this point travelling on the loop, the hierarchy level changes from 1 to 3.

An immediate consequence of Definition 8 is:

Proposition 1. *Every strange point is either an increasing strange point or a decreasing strange point and this implies that every point on a path α is either:*

- *An increasing strange point*
- *A decreasing strange point*
- *Not a strange point*

Definition 9. *Let $S_{\alpha,f} \coloneqq \{\alpha(t) \mid t \in [0,1]$ and $\alpha(t)$ is a strange point$\}$ be the multiset of strange points. Let the function g_f be defined as:*

$$g_f(\alpha(t)) \coloneqq \begin{cases} +1 & \text{if } \alpha(t) \text{ is an increasing strange point} \\ 0 & \text{if } \alpha(t) \text{ is not a strange point} \\ -1 & \text{if } \alpha(t) \text{ is a decreasing strange point} \end{cases}$$

If $S_{\alpha,f}$ is finite, then the <u>*strangeness index*</u> $n_{\alpha,f}$ *is defined as:*

$$n_{\alpha,f} \coloneqq \sum_{\alpha(t) \in S_{\alpha,f}} g_f(\alpha(t))$$

Note: By Proposition 1, the function g_f is well-defined for all points on the path.

The strangeness index is defined to capture a 'net strangeness' of a path and it is based on how many times a path passes through a strange point and whether there was an increase or decrease in the hierarchy level when passing through this strange point.

The reason that $S_{\alpha,f}$ is a multiset rather than a set is that a path could pass through a strange point multiple times. For instance, consider a loop that passes through one strange point. If one were to traverse this loop twice, then one would pass through this strange point twice. So this strange point should contribute to the strangeness index twice, hence there should be two copies of this point in $S_{\alpha,f}$.

Note: Two copies of the same strange point in $S_{\alpha,f}$ can be an increasing strange point and a decreasing strange point. For example, consider a figure eight loop with a strange point at this loop's intersection. As one traverses around this loop, the first time passing through this strange point, there is an increase/decrease in hierarchy levels. The second time passing through this strange point, there is a decrease/increase in hierarchy levels. Hence one copy of this strange point in $S_{\alpha,f}$ is an increasing strange point and the other copy is a decreasing strange point.

Definition 10. *Let $\alpha : [0,1] \to \mathbb{C}$ be a loop (path with the same start and end point), then α is a* <u>*(generalised) strange loop (on f)*</u> *if the strangeness index $n_{\alpha,f}$ is not equal to 0.*

In other words, a loop is a strange loop if the number of increasing strange points in $S_{\alpha,f}$ is not equal to the number of decreasing strange points in $S_{\alpha,f}$.

For example, the red loop in Figure 17 is not a strange loop since this loop has one increasing and one decreasing strange point. In addition, the orange loop in Figure 18 is a strange loop. The orange loop has one increasing strange point and two decreasing strange points, so this loop has a strangeness index of -1 and is therefore a strange loop.

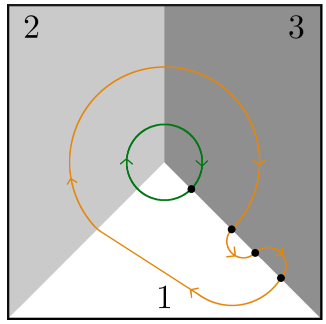

Figure 18: Strange loops with strange points indicated by solid black circles

Definition 11. *Let α be a strange loop and let $n_{\alpha,f}$, $S_{\alpha,f}$ be defined as in Definition 9, then:*

- *α is a strong strange loop (on f) if $|n_{\alpha,f}| = |S_{\alpha,f}|$*
- *α is a weak strange loop (on f) if $|n_{\alpha,f}| \neq |S_{\alpha,f}|$*

Proposition 2. *Let α be a strange loop, then the following are true:*

1. *α is a strong strange loop $\iff$ α contains only increasing strange points or contains only decreasing strange points.*
2. *α is a weak strange loop $\iff$ α contains a mixture of increasing strange points and decreasing strange points.*

Proof. Assume that the multiset $S_{\alpha,f}$ has m increasing strange points and n decreasing strange points where $m, n \in \mathbb{N}_0$, this implies that $|S_{\alpha,f}| = m+n$ and $|n_{\alpha,f}| = |m-n|$. Since α is a strange loop, this implies that $m \neq n$.

1. Hence α is a strong strange loop $\iff$ $|n_{\alpha,f}| = |S_{\alpha,f}|$ $\iff$ $|m-n| = m+n$ $\iff$ $m = 0$ or $n = 0$

   Since $m \neq n$, this implies that the condition $m = 0$ or $n = 0$ is equivalent to α containing only increasing strange points ($n = 0$) or containing only decreasing strange points ($m = 0$).

2. Hence α is a weak strange loop $\iff$ $|n_{\alpha,f}| \neq |S_{\alpha,f}|$ $\iff$ $|m-n| \neq m+n$

   There are two cases:

   - If $m > n$, then $|m-n| \neq m+n \iff n \neq 0 \iff m > n > 0$
   - If $m < n$, then $|m-n| \neq m+n \iff m \neq 0 \iff n > m > 0$

   Both of these cases are equivalent to α containing a mixture of increasing strange points and decreasing strange points (both m and n are strictly greater than 0).

□

Using Proposition 2, the green loop in Figure 18 is a strong strange loop since this loop has one decreasing and no increasing strange points. However, the orange loop in Figure 18 is a weak strange loop since this loop has two decreasing and one increasing strange point.

How is Hofstadter's definition of a strange loop linked to Defintion 10? Consider Hofstadter's definition of a strange loop. This definition describes a loop that passes through an decreasing/increasing strange point to return to the starting level. This means the strangeness index of this loop is ∓ 1 and therefore this loop is a strange loop. In addition, this loop passes through one decreasing/increasing strange point and passes through no increasing/decreasing strange points. Hence, the strange loops in Hofstadter's definition are strong strange loops.

The term *generalised* strange loop is used in Definition 10 since weak strange loops do not match the definition of a strange loop given in Hofstadter's definition. However these loops do exhibit a 'net strangeness' (the number of increasing and decreasing strange points are not equal).

5 Strange Loops and Tiling Hierarchies

In Section 4, the researcher demonstrated how a strange loop could potentially be produced on Escher's Grid (Figure 12) or some variation of Escher's Grid. In this section, the researcher will present a method for dividing a substitution tiling into different hierarchy levels such that strange loops are produced in variations of Escher's Grid. This was first done by McCoyd for Escher's Grid [6], however the researcher will develop an independent method that can be adapted for variations of Escher's Grid. The researcher will then apply this method to produce a new visual of a distorted square chair tiling.

5.1 Constructing a Strange Loop of Tiling Hierarchies

Recall from Section 2.1 that the bijection h_x^{-1} maps an undistorted grid invariant under scalar multiplication by x to a distorted grid (a variation of Escher's Grid) invariant under complex multiplication by γ_x. So for an undistorted self-similar substitution tiling T produced by a self-similar substitution map σ to be mapped to a variation of Escher's Grid, it is required that T is a fixed point of σ (that is $\sigma(T) = T$) and that $x = \lambda^m$ where λ is the expansion factor of σ and $m \in \mathbb{N}$.

Next, one must figure out how to divide T so that different hierarchy levels are displayed on this tiling. By Remark 4, this implies that some tiles in T are replaced by their corresponding supertiles level n scaled down by factor λ^n for some $n \in \mathbb{N}$ (this fully replaces the original tile since σ is a self-similar map).

A *supertile line* can be defined as a line on a tiling that follows the boundaries of the tiles such that when crossing this line, tiles are replaced with their corresponding supertiles level n for some $n \in \mathbb{N}$. The supertile line is defined to follow the boundaries of the tiles to preserve the overall tiling structure. For example, given a substitution tiling generated by square prototiles of equal length, Figure 19 represents a hierarchy level template for this substitution tiling with supertile lines at the boundaries between two colours (note that the supertile lines will follow the boundaries of the square tiles).

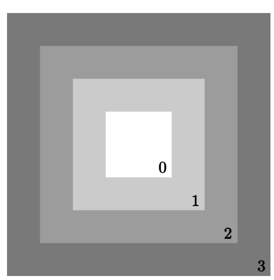

Figure 19: Sketch hierarchy template on the undistorted tiling generated by square prototiles (numbers represents the hierarchy level/supertile level)

The remainder of this report will consider a self-similar square tiling T generated by a substitution map σ with square prototiles of equal length and the expansion factor of this substitution map is λ. Figure 19 gives a hierarchy template for square prototiles. So each tile in the region labelled n (where $n \in \{1, 2, 3\}$) is replaced with its corresponding supertile level n scaled down by a factor of λ^n.

This hierarchy template can be used to define a hierarchy function for the entire tiling. Assume that Figure 19 was constructed so that the supertile line between level 0 and 1 is similar to the outer line of level 3 by a factor of $x = \lambda^m$ where $m \in \mathbb{N}$. Scale up this hierarchy template by factor x. The scaled up version's level 0 square matches with the original dimensions of Figure 19, so now replace this scaled up level 0 square with the hierarchy template in Figure 19. The result of this is given in Figure 20. By repeating this process indefinitely, one can generate the hierarchy function of the entire tiling. By imposing this hierarchy function onto the tiling, this has created an undistorted tiling that contains multiple hierarchy levels.

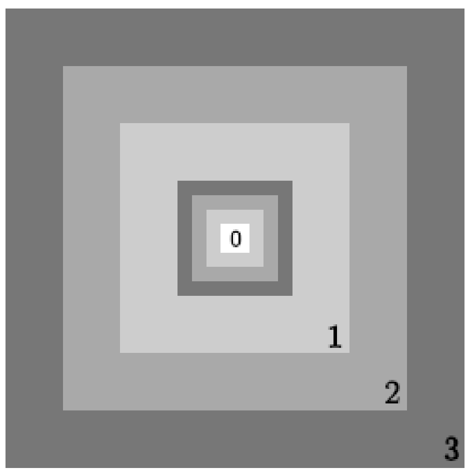

Figure 20: Sketch result of scaling Figure 19 (colours represents the hierarchy level)

By construction, all of the level 1, 2, 3 square annuli of the hierarchy function generated in the previous method are similar to each by scale factor x. Therefore this method has created an undistorted tiling that contains multiple hierarchy levels where the levels of the hierarchy are invariant under scalar multiplication by x. So how is this tiling mapped under the function h_x^{-1} to a variation of Escher's Grid?

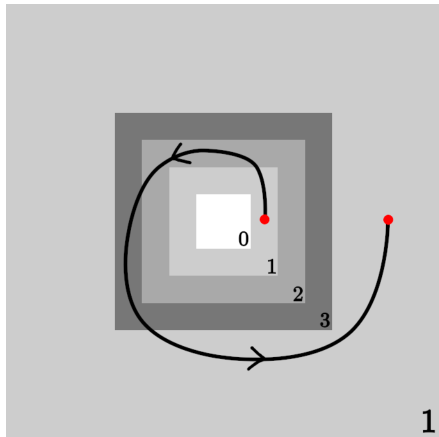

Figure 21: Anticlockwise path on Figure 20 where the start and end points differ by a factor of x

Consider the path in Figure 21. Recall from Section 2.1 that the map h_x^{-1} takes a single anticlockwise path around the origin where the start and end points are similar by a scale factor of x in the undistorted grid and maps it to a single anticlockwise loop around the origin in the variation of Escher's Grid. Since the path in Figure 21 is a path on the undistorted grid where its start and end points are similar by a scale factor of x, then this path is a single anticlockwise loop in the distorted grid (the variation of Escher's Grid). Is this loop a strange loop in the distorted grid? As one traverses the path on the undistorted tiling in Figure 21, the hierarchy levels of the tiling changes in the following way:

$$1 \to 2 \to 3 \to 1$$

So this path has only one strange point and it is located at the point where the hierarchy level on the path changes from 3 to 1. This implies that this point is a decreasing strange point and hence the strangeness index of this path on this hierarchy function is -1. When the undistorted tiling is transformed under a given function, the hierarchy function is also transformed under the same function. Therefore, this path's strangeness index is invariant under h_x^{-1} which means this transformed path (which is now a single anticlockwise loop) is a strange loop in the variation of Escher's Grid.

A method has now been provided on how to divide an undistorted square tiling into different hierarchy levels so that strange loops are produced in a variation of Escher's Grid.

Remark 5. Consider a supertile line between level 1 and level 2, then there are supertiles level 1 and supertiles level 2 that have a boundary that intersects on this supertile line. So which hierarchy level does this boundary belong to? For this research, if a supertile level i and a supertile level j share a boundary where $i > j$, then assume that the boundary/supertile line has a hierarchy/supertile level of i. This allows for strange points to be defined on sections of paths that are on and parallel to supertile lines.

Consider the loops in Figure 22. When the loops are on and parallel to the supertile line, the hierarchy/supertile level of this line is 3. This means the blue loop is contained entirely in level 3 so this loop has no strange points. However, all the sections of the red loop that are not on the supertile line belong to level 1. This means there are 2 strange points on this loop (indicated by black circles) where the lowest strange point is an increasing strange point and the highest strange point is a decreasing strange point.

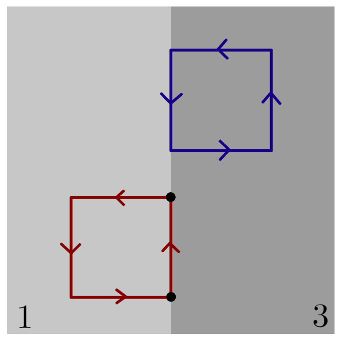

Figure 22: Loops with sections that are parallel to the supertile line between level 1 and 3 with strange points indicated by black circles

5.2 Escher's Square Chair Tiling

In this subsection, the researcher will use the method from the previous subsection to construct a square chair tiling containing multiple hierarchy levels imposed on a variation of Escher's Grid such that strange loops are produced. The square chair tiling is a tiling generated by a substitution map σ and how σ acts on each of the prototiles is given in Figure 23. Figure 23 shows that the expansion factor of the square chair tiling is $\lambda = 2$.

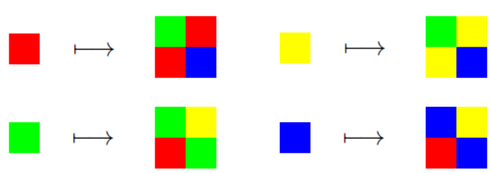

Figure 23: The substitution map σ for the square chair tiling

Consider the supertile level 1 $\sigma(Red)$. This supertile defines a tiling on the plane by repeatedly applying σ to $\sigma(Red)$, with the origin defined where the four tiles of $\sigma(Red)$ intersect. Figure 24 is generated by applying this iteration to $\sigma(Red)$ two times (which is equivalent to the supertile level 3 generated by the red prototile). Figure 25 will be used as the hierarchy template for Figure 24.

From section 5.1, it is required that in Figure 25 that the supertile line between level 0 and level 1 is similar to the outer line for level 3. Since Figure 25 is to scale, this implies that $x = 2^2$. Now apply the methodology in the section 5.1 to generate a square chair tiling that contains different hierarchy levels on this tiling. Figure 28 is a subset of this tiling which is equivalent to imposing Figure 27 onto Figure 26.

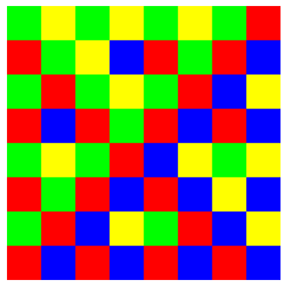

Figure 24: Supertile level 3 generated by the red prototile

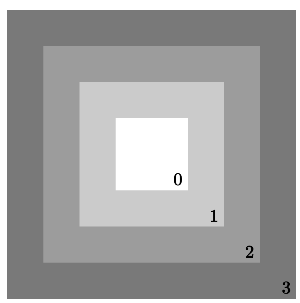

Figure 25: Hierarchy template (to scale) for Figure 24

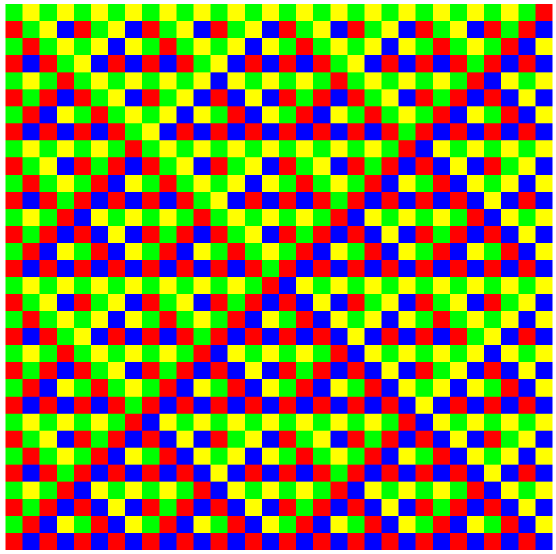

Figure 26: Supertile level 5 generated by the red prototile/σ acting on $\sigma(Red)$ four times

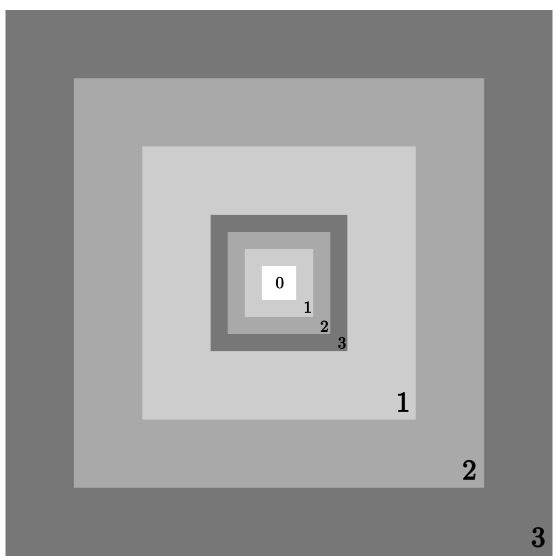

Figure 27: Hierarchy function to be imposed on Figure 26

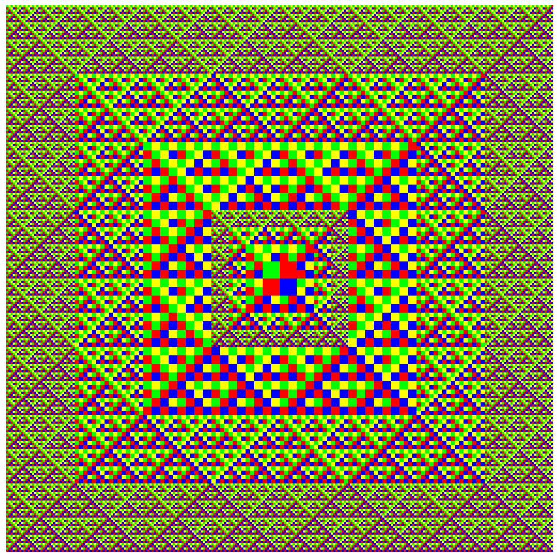

Figure 28: Result of imposing Figure 27 on Figure 26

Figure 28 is a subsection of the square chair tiling on an undistorted grid where the regions of hierarchy levels/supertile levels are similar to each other by a scale factor of $x = 4$. So what does Figure 28 look like under the transformation h_4^{-1}?

To do this, the researcher used the software Mathematica and code provided in [7] to transform images under complex conformal transformations. The result is given in Figure 29.

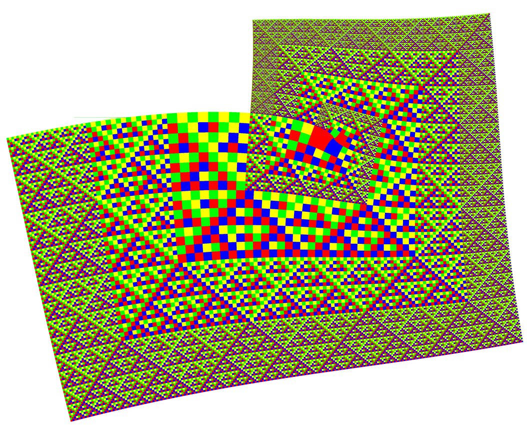

Figure 29: Result of transforming Figure 28 under h_4^{-1}

Then we obtain *Escher's Square Chair Tiling* (Figure 30) by zooming in to the centre of the supertile level 0 tiles (the original prototiles) on Figure 29 and removing any lines that were produced in error by the Mathematica code in [7].

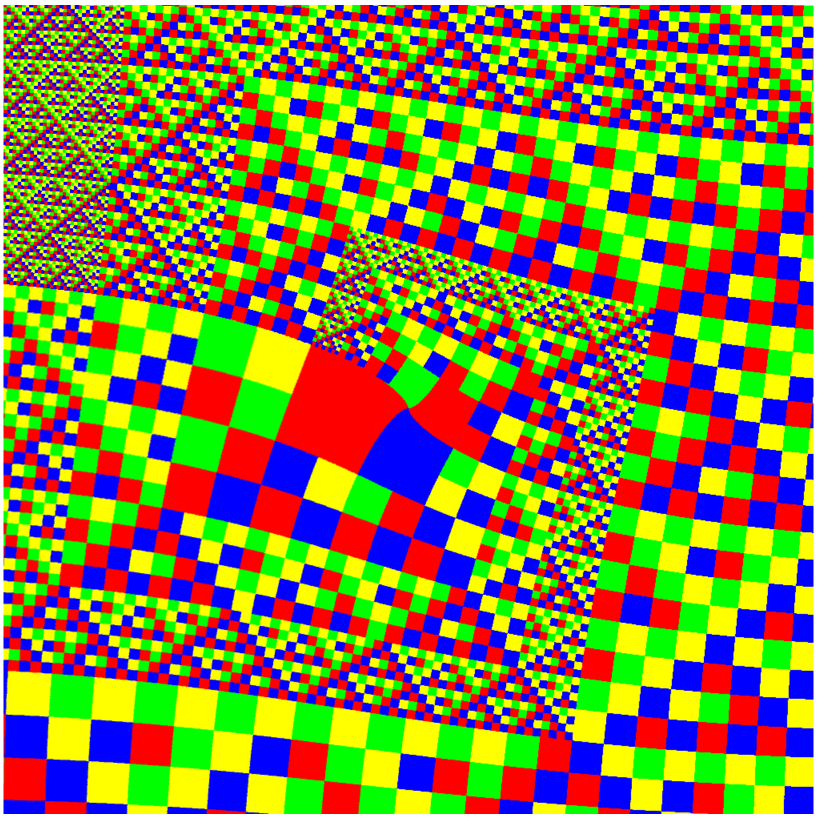

Figure 30: Escher's Square Chair Tiling

The method provided in Section 5.1 demonstrates that there are single anticlockwise loops in Escher's Square Chair Tiling such that these loops are strange loops. However, there is a more general result for single anticlockwise loops:

Proposition 3. *Let α be a single anticlockwise loop on Escher's Square Chair Tiling that does not pass through the hierarchy level* 0*, then the strangeness index of α is* -1*.*

Sketch Proof. Since α is a single anticlockwise loop in the variation of Escher's Grid, then transforming this loop under h_4 back onto the undistorted grid gives a new path β where β is an anticlockwise path around the origin where the start and end point is similar by a scale factor of 4 (and the start point is closer to the origin than the end point).

Since Figure 25 was used as a hierarchy template to define the hierarchy function for the undistorted tiling, this means the start and end point of β

belong to the same hierarchy level (let us say that the start point belongs to the *inner* annulus and the end point belongs to the *outer* annulus). In order for the path β to reach the end point, this path must pass through at least one decreasing strange point. The blue point in Figure 31 illustrates this. This gives a contribution of -1 to the strangeness index.

The path β could travel radially outwards/inwards from the outer annulus and pass through more decreasing/increasing strange points. However, for the path to return to the end point in the outer annulus, each decreasing/increasing strange point must have a corresponding increasing/decreasing strange point. The red points on the solid path and dashed path in Figure 31 give an illustration of this. This gives a contribution to the strangeness index of 0.

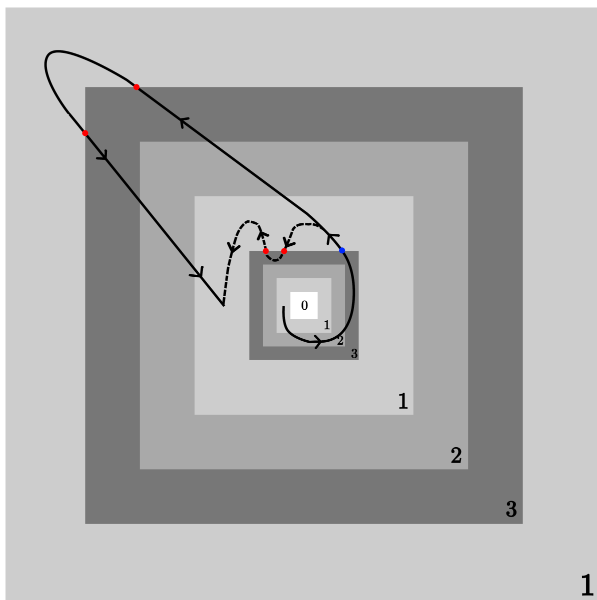

Figure 31: Solid and dashed path on the undistorted tiling's hierarchy function (the dashed path is a branch of the solid path)

Hence, the overall contribution to the strangeness index in the undistorted tiling is -1 and applying h_4^{-1} to β to give the single anticlockwise loop α in Escher's Square Chair Tiling implies that the strangeness index of α is -1. $\square$

This has shown that any single anticlockwise loop (that does not pass through hierarchy level 0) in Escher's Square Chair Tiling is a strange loop between the tiling hierarchy levels. Some examples of single anticlockwise loops on Escher's Square Chair Tiling is given in Figure 32. The solid loop is a strong strange loop since if one follows this loop's corresponding path in the undistorted tiling in Figure 32, there is one decreasing strange point and no increasing strange points.

The dashed loop (which is a branch of the solid loop) on Escher's Square Chair Tiling in Figure 32 is a weak strange loop since if one follows this loop's corresponding path in the undistorted tiling in Figure 32, there are two decreasing strange points and one increasing strange point. Both loops have a strangeness index of -1 which verifies Proposition 3.

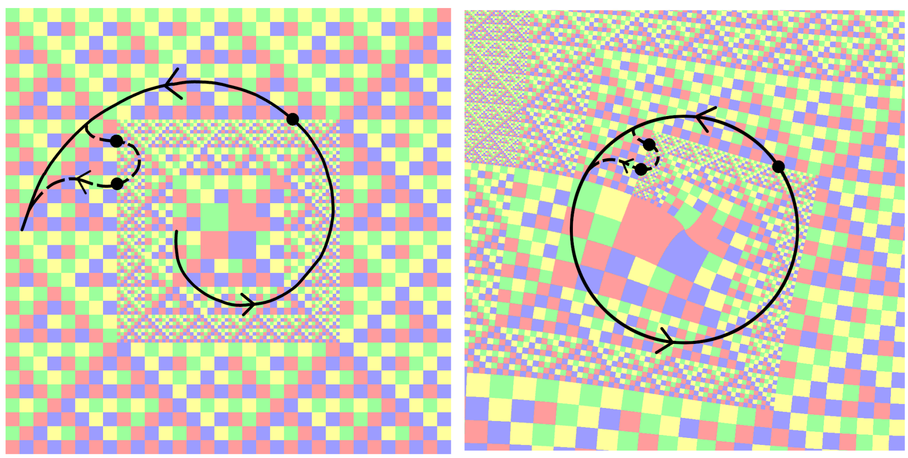

Figure 32: Solid and dashed anticlockwise loop on Escher's Square Chair Tiling (Right) and their corresponding paths in the undistorted tiling (Left) (Strange points indicated by black circles)

Remark 6. Consider any anticlockwise loop α on Escher's Square Chair Tiling that does not pass through hierarchy level 0. Assume that α has i increasing strange points (counted with multiplicity), then by Proposition 3, α has $i+1$ decreasing strange points (counted with multiplicity). If one traverses this loop n times where $n \in \mathbb{N}$, then this new loop has ni increasing strange points and $n(i+1)$ decreasing strange points (counted with multiplicity).

Hence by the definition of the strangeness index, the strangeness index of this new loop is $-n$. This shows there may be a link between the winding number around the origin and the strangeness index for this particular hierarchy function.

6 Conclusions and Further Research

This research has provided a method for representing undistorted substitution tilings that contain multiple hierarchy/supertile levels such that strange loops are produced between these hierarchy levels on variations of Escher's Grid, and this research has produced the visual *Escher's Square Chair Tiling* (Figure 30) using this method.

The method provided only examined substitution tilings that were generated by square prototiles of equal length. This method could potentially be adapted to include the following types of substitution tilings:

- Tilings where the prototiles have the same geometric shape but are not squares (i.e. hexagons, triangles etc.)
- Tilings where the prototiles have different geometric shapes (i.e. Penrose Tiling)

However, more visuals can be produced using the current method by considering the following:

- Changing the hierarchy template used to define the hierarchy function on the undistorted tiling to include more hierarchy/supertile levels by adding more supertile lines.
- Using different substitution tilings where the prototiles are squares of equal length (i.e. 2D Thue-Morse Tiling)

A problem with the image Escher's Square Chair Tiling was in order to demonstrate that strange loops are produced in this image, it was required to examine their corresponding paths in the original undistorted tiling (see Figure 32). So is there a way to indicate that strange loops are produced in Escher's Square Chair Tiling by looking at *only* Escher's Square Chair Tiling?

In this research, h_x^{-1} (the transformation that mapped the undistorted tiling to a distorted tiling) was constructed to reflect a mathematical interpretation of a decision Escher made, which is discussed in Remark 3. In addition, Remark 3 discusses the possibility of changing where the anticlockwise loop in the distorted grid's fundamental group is mapped to in the undistorted grid's fundamental group. So one could change this analysis to map all anticlockwise loops in a distorted grid to a different element of the undistorted grid's fundamental group. This would change the complex geometry of the distorted grid and one could adapt the method in Section 5.1 to produce strange loops on these new complex geometries.

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Appendix A - Proof of h_x^{-1}

Remark. *For the purposes of this appendix, the functions* h_x *and* h_x^{-1} *will be written instead as* h *and* h^{-1} *respectively. In addition,* α_x *and* γ_x *will be written instead as* α *and* γ *respectively.*

Claim 1. $log(e^z) = z + 2\pi ik$ *for some* $k \in \mathbb{Z},\ z \in \mathbb{C}$

Proof. The definition of the complex logarithm is: $log(e^z) = log|e^z| + iarg(e^z)$ where arg is an arbitrary choice of argument.

Since $|e^z| = e^{Re(z)}$ and $arg(e^z) = arg(e^{Re(z)}e^{iIm(z)}) = arg(e^{iIm(z)}) = Im(z) + 2\pi k$ (where $k \in \mathbb{Z}$ is dependent on the choice of argument), $log(e^z)$ is now given by:

$log(e^z) = log(e^{Re(z)}) + i(Im(z) + 2\pi k) = Re(z) + iIm(z) + 2\pi ik = z + 2\pi ik$ □

Claim 2. $\alpha = 2\pi i/(2\pi i(n+1) - log\gamma)$ *for some* $n \in \mathbb{Z}$

Proof. Since $\gamma = e^{a^{-1}logx}$ and by using Claim 1 gives:

$$log\gamma = \alpha^{-1}logx + 2\pi in \text{ for some } n \in \mathbb{Z}$$

Rearranging this equation for $logx$ gives:

$$logx = \alpha(log\gamma - 2\pi in) \tag{1}$$

Since $\alpha = (2\pi i + logx)/2\pi i$, by substituting 1 into this expression gives:

$$\alpha = (2\pi i + \alpha(log\gamma - 2\pi in))/2\pi i$$

By rearranging the above equation for α, this gives the result stated in Claim 2. □

So now there are two expressions for α, namely $\alpha = (2\pi i + logx)/2\pi i$ and $\alpha = 2\pi i/(2\pi i(n+1) - log\gamma)$ for some $n \in \mathbb{Z}$.

Claim 3. $h^{-1}(w) = e^{\alpha^{-1} logw}$ *is the inverse function of* $h(w) = e^{\alpha logw}$.

Proof. The inverse of h is a bijection from $\mathbb{C}^*/{<}x{>} \to \mathbb{C}^*/{<}\gamma{>}$, which means that equivalence classes of complex numbers must be considered under the maps $h \circ h^{-1}$ and $h^{-1} \circ h$.

Consider the complex number $w \cdot x^m$ for some $w \in \mathbb{C}^*$ and $m \in \mathbb{Z}$. This complex number belongs to the equivalence class $w{<}x{>}$. Hence:

$$h \circ h^{-1}(w{<}x{>}) = h \circ h^{-1}(w \cdot x^m) = h(e^{\alpha^{-1} log(w \cdot x^m)}) = e^{\alpha log(e^{\alpha^{-1} log(w \cdot x^m)})}$$

Using Claim 1, the above expression for $h \circ h^{-1}$ becomes:

$$h \circ h^{-1}(w{<}x{>}) = e^{\alpha(\alpha^{-1} log(w \cdot x^m) + 2\pi ik)} = e^{log(w \cdot x^m)} e^{2\pi \alpha ki} = w \cdot x^m e^{2\pi \alpha ki}$$

for some $k \in \mathbb{Z}$.

Substituting in $\alpha = (2\pi i + logx)/2\pi i$ gives:

$$h \circ h^{-1}(w{<}x{>}) = w \cdot x^m e^{2\pi ki((2\pi i + logx)/2\pi i)} = w \cdot x^m e^{k(2\pi i + logx)}$$

Since $k \in \mathbb{Z}$, this implies that $e^{2\pi ik} = 1$ and the above becomes:

$$h \circ h^{-1}(w{<}x{>}) = w \cdot x^m e^{2\pi ik} e^{klogx} = w \cdot x^m x^k = w \cdot x^{m+k}$$

The complex number $w \cdot x^{m+k}$ belongs to the equivalence class $w{<}x{>}$. Hence, the above shows that:

$$h \circ h^{-1}(w{<}x{>}) = w{<}x{>} \iff h \circ h^{-1} = Id \qquad (\dagger)$$

which means that h^{-1} is the right inverse to h.

Now consider the complex number $w\gamma^m$ for some $w \in \mathbb{C}^*$ and $m \in \mathbb{Z}$. This complex number belongs to the equivalence class $w{<}\gamma{>}$. Hence:

$$h^{-1} \circ h(w{<}\gamma{>}) = h^{-1} \circ h(w\gamma^m) = h^{-1}(e^{\alpha log(w\gamma^m)}) = e^{\alpha^{-1} log(e^{\alpha log(w\gamma^m)})}$$

Using Claim 1, the above expression for $h^{-1} \circ h$ becomes:

$$h^{-1} \circ h(w{<}\gamma{>}) = e^{\alpha^{-1}(\alpha log(w\gamma^m) + 2\pi ik)} = e^{log(w\gamma^m)} e^{2\pi \alpha^{-1} ki} = w\gamma^m e^{2\pi \alpha^{-1} ki}$$

for some $k \in \mathbb{Z}$.

Claim 2 implies that α^{-1} can be represented as $\alpha^{-1} = (2\pi i(n+1) - log\gamma)/2\pi i$ for some $n \in \mathbb{Z}$ and substituting α^{-1} into the above gives:

$$h^{-1} \circ h(w{<}\gamma{>}) = w\gamma^m e^{2\pi ki((2\pi i(n+1) - log\gamma)/2\pi i)} = w\gamma^m e^{k(2\pi i(n+1) - log\gamma)}$$

Since $k(n+1) \in \mathbb{Z}$, this implies that $e^{2\pi ik(n+1)} = 1$ and the above becomes:

$$h^{-1} \circ h(w\!<\!\gamma\!>) = w\gamma^m e^{2\pi ik(n+1)} e^{-klog\gamma} = w\gamma^m e^{-klog\gamma}$$

The definition of a complex power implies that $\gamma^m = e^{mlog\gamma}$ and substituting this into the above gives:

$$h^{-1} \circ h(w\!<\!\gamma\!>) = we^{mlog\gamma} e^{-klog\gamma} = we^{(m-k)log\gamma} = w\gamma^{m-k}$$

The complex number $w\gamma^{m-k}$ belongs to the equivalence class $w\!<\!\gamma\!>$. Hence, the above shows that:

$$h^{-1} \circ h(w\!<\!\gamma\!>) = w\!<\!\gamma\!> \iff h^{-1} \circ h = Id \qquad (\dagger\dagger)$$

which means that h^{-1} is the left inverse to h.

Combining † and †† shows that h^{-1} is both the left and right inverse of h which implies that h^{-1} is the inverse function of h. □