The localization transition in the sub-Ohmic spin-boson model: a short review

Riccardo Bonzano*
Laidlaw research project, supervised by Dr Brendon Lovett*
**Supa, School of Physics and Astronomy, University of St Andrews,*
North Haugh, St Andrews, Fife KY16 9SS, UK

(Dated: June 2020)

1 Dissipative quantum phase transitions

A quantum phase transition (QPT) is a phase transition which occurs at zero temperature in response to the variation of some non-thermal parameter (e.g., pressure, magnetic field). It is caused by quantum fluctuations subsequent to Heisenberg's uncertainty principle and it takes the system between two distinct ground states [1, 2]. Let's consider a system at $T = 0$ which is open (i.e., it interacts with an external quantum system, called the environment or bath) and which experiences dissipation. This dissipative coupling to the environment impacts considerably on the system dynamics and can trigger interesting phase transitions: in this case, we talk about dissipative quantum phase transitions [3].

2 The spin-boson model (SBM)

In a seminal paper in 1981, A. Caldeira and A. J. Leggett proposed a model, known as the Caldeira– Leggett or harmonic bath model, to investigate quantum dissipation [4]. It describes a quantum particle in one dimension coupled to a bath modelled by an infinite sum of harmonic oscillators [5]; this method was first proposed by Feynman and Vernon in 1963 [6].

The “dissipative two-level system” (DTS) is a particular realization of the Caldeira–Leggett model [7]. It describes the effects of dissipation on the dynamics of a particle that is only allowed to assume two different positions instead of being free to slide on a continuous coordinate. For example, the system might be a quantum particle delocalized in a double well potential. Within this framework, it is possible to investigate one of the simplest and most studied dissipative QPT: the localization transition of a quantum particle by the bath. When the interaction with the bath is strong enough, quantum tunneling is suppressed, the wavefunction is collapsed and the particle is localized in either one of the two wells [7]. Intuitively, any quantum mechanical behaviour by the system is destroyed when the environment “looks at it” [8].

The DTS has a reduced two-dimensional Hilbert space, which allows the problem to be described in terms of 1/2 spin operators. Since it is an open quantum system, it can be described by the tripartite Hamiltonian

$$H = H_{system}(\psi) + H_{bath}(\eta) + H_{int}(\psi, \eta), \tag{1}$$

where ψ, η characterize respectively the state of the system and the bath, and where H_{int} accounts for the system-bath interaction. For completeness, we include below the famous form by Leggett et al [7]:

$$H = -\frac{1}{2}\hbar\Delta\sigma_x + \frac{1}{2}\epsilon\sigma_z + \sum_\alpha \left(\frac{1}{2}m_\alpha\omega_\alpha {x_\alpha}^2 + \frac{{p_\alpha}^2}{2m_\alpha}\right) + \frac{1}{2}q_o\sigma_z \sum_\alpha C_\alpha x_\alpha. \tag{2}$$

H_{system} is expressed by the first two terms, where the first accounts for tunneling and the second for the energy of the two wells singularly. The third term encodes H_{bath} by accounting for the infinite sum of harmonic oscillators that models the bath, while the last term describes H_{int}. This particular form of the Hamiltonian, known as the spin-boson Hamiltonian, contributed to the name that the DTS has acquired in the literature: spin-boson model (SBM). The SBM is the simplest non-trivial model that describes dissipation in a quantum mechanical system [9].

2.1 The spectral function and the -*Ohmic* cases

The bath spectral function encapsulates the complete effect of the environment. Again for completedness, we include the form by Leggett et al [7]

$$J(\omega) = \frac{\pi}{2} \sum_{\alpha} \left(\frac{{C_\alpha}^2}{m_\alpha \omega_\alpha} \right) \delta(\omega - \omega_\alpha). \tag{3}$$

It is assumed to be a smooth function of ω and to display a simple power-law behaviour $J(\omega) = A\omega^s$ up to some cutoff frequency ω_c. The case for which $s = 1$ is called *Ohmic*, $0 < s < 1$ *sub-Ohmic* and $s > 1$ *super-Ohmic* (the case $s < 0$ is pathological) [7]. In the Ohmic case, it is a consensus in the literature that the phase transition from delocalization to localization (a Kosterlitz-Thouless transition) happens with the increase of the coupling strength between the system and the bath. This is due to the competition between the internal tunneling effect of the system and the external dissipation of the environment. In this review we will focus on the more involved and debated phase transition in the sub-Ohmic case, giving a brief overview of the state-of-the-art of the field.

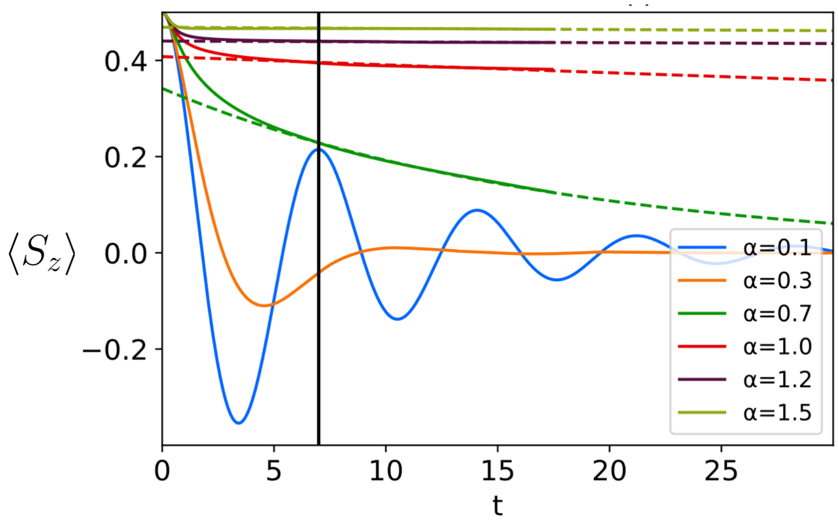

Figure 1: This graph shows the localization transition of the Ohmic SBM. The transition occurs at critical value of the coupling strength $\alpha_c = 1.25$. The system is set up in $\langle S_z \rangle = 1/2$, with no environment excitation. In the steady state, below α_c the system does not localize (tunneling is not suppressed) and decays to $\langle S_z \rangle = 0$; above α_c it localizes in the state it started in. For $0.5 < \alpha < \alpha_c$, the system shows incoherent exponential decay. This graph was produced by exploiting the TEMPO algorithm [10]. Own figure.

2.2 Why is the SBM interesting?

The SBM is especially interesting due to its potential for the field of quantum computation [11, 12], as the decoherence of this system as a qubit is a major hindrance to the practical realization of quantum information processing [13]. Furthermore, it displays incredibly rich physics in quantum criticality and decoherence [14, 15].

3 The sub-Ohmic SBM

3.1 The localization transition

In their seminal paper "Dynamics of the dissipative two state system" [7], Leggett et al used the popular non-interacting blip approximation to analyse the sub-Ohmic unbiased (i.e., no detuning between the two wells) case. They found that tunneling is suppressed at zero temperature and no QPT occurs, while at finite temperatures the system shows an exponential overdamped relaxation rate proportional to $\exp(-(T_0/T)^{1-s})$. Today, however, it is believed that the QPT is present, although not visible if using the non-interacting blip approximation. In other words, as the coupling strength between the system and the environment increases, a transition from weakly damped coherent motion to localization occurs in both the Ohmic and sub-Ohmic cases. This result was obtained in a number of papers via different methods, as detailed below.

A line of continuous QPT with associated critical behavior was found for all $0 < s < 1$ using the non-perturbative numerical renormalisation group method (NRG) [16]. By expanding on this work, it was demonstrated that, in contrast to the Ohmic case, the sub-Ohmic delocalized phase cannot be characterised by a single energy scale only, due to the fact that the QPT is non-trivial. It was shown that perturbative methods [17], since they only capture the renormalized coherent tunneling, give sensible results at short times, but fail in the long time limit. Furthermore, in the strongly sub-Ohmic regime with $s \ll 1$, it was observed that weakly damped coherent oscillations on short time scales are possible even in the localized phase; this is of crucial relevance to model qubits subject to electromagnetic noise [18].

The sub-Ohmic QPT was also found by using: the perturbation approach based on a unitary transformation [19]; an extended hierarchy equation of motion [20]; the density matrix renormalization group approach [21]; the numerically exact multilayer multiconfiguration time-dependent Hartree (ML-MCTDH) method [22]; the flow equations approach [23]. The use of NRG in combination to hyperscaling relations confirmed the above results and found that the entanglement between the spin and its environment is always enhanced at the phase transition, resulting in a cusp (maximum) in the entropy of entanglement [14]. This leads to a rigorous unification between entanglement of the spin with its bath, decoherence, and QPT [24]. The extension of NRG to the Bose-Fermi Kondo model found, for sub-Ohmic bosonic bath exponents, critical properties that are identical to those of the SBM [25]. A general extended coherent state approach was applied successfully and the QPT was located by finding the minimum of the ground state fidelity [26]. A quantum simulation by color centers in free-standing hexagonal boron nitride (hBN) membranes was proposed and a property peculiar to the strongly sub-Ohmic QPT was found [27]. Real-time path integral Monte Carlo (PIMC) techniques were used to show that large coupling strength does not induce incoherent relaxation for spectral exponents $0 < s < 1/2$ even when the thermal equilibrium is almost classical [28]. Finally, P. Nalbach and M. Thorwart studied the ultraslow quantum dynamics associated with the sub-Ohmic bath and obtained an excellent phase diagram, displayed below. [29].

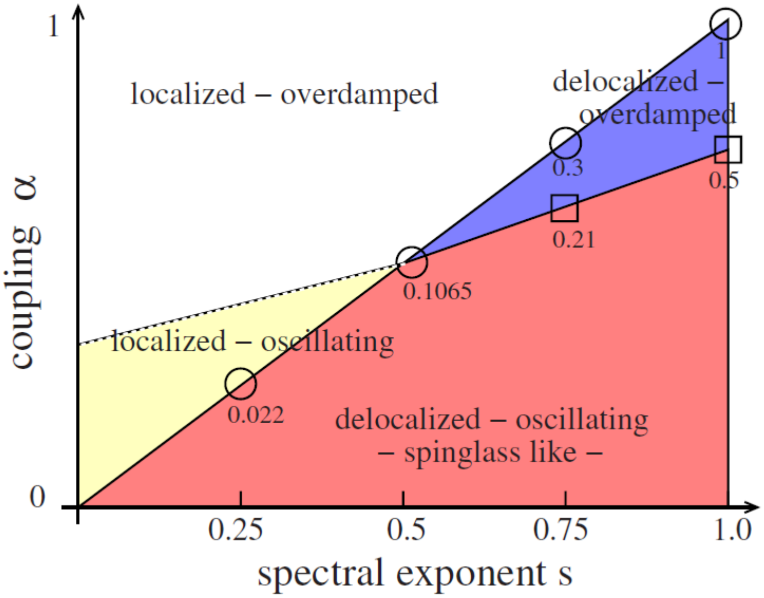

Figure 2: Sub-Ohmic SBM phase diagram at T=0. This image was adapted from [29].

3.2 The quantum to classical mapping

Similarities between models can be an extremely powerful analytical tool as overly complex features of one model might be clarified by clearer features of another. A procedure, known as the quantum to classical mapping (QCM), makes it possible to map the partition function of a n-dimensional quantum system to the partition function of a $n+1$-dimensional classical system [30]. It can be exploited to investigate quantum phase transitions by analysing their classical counterparts. Additionally, by using Monte Carlo simulations of relevant classical, the QCM becomes a powerful tool for the study of new quantum universality classes [30, 31].

Under this mapping, the sub-Ohmic SBM is equivalent to a classical Ising spin chain with long-range interactions. Consequently, it is expected to display a continuous magnetic transition with mean-field critical exponents. Vojta et al seemed to disprove the above when, by using NRG, they found the transition to be characterised by non-mean-field critical properties for $0 < s < 1/2$ [32]. The result was supported by [33, 34].

However, a few years later Vojta et al reported an error in the previous publication: they showed that there is no breakdown of the QCM and that the exponents are mean-field [35, 36]. A number of subsequent articles confirmed the result by using several different techniques: quantum monte carlo technique [37]; sparse polynomial approach [38]; a variational ansatz for the ground state [39, 40]; extended coherent approach [26].

3.3 Why is the sub-Ohmic QPT interesting?

The interest for the localization transition of the SBM with sub-Ohmic spectral density is not purely academic. It has been used to model the 1/f noise [41] in quantum dots [42, 43] and superconductor qubit systems [44, 45]. It also appears in the context of ultraslow glass dynamics [46], quantum impurity systems [47, 48], and nanomechanical oscillators [49]. The sub-Ohmic case of $s = 1/2$ can be used to describe electromagnetic transmission lines [50]. Furthermore, $s = 1/2$ baths may be realized in the context of effective impurities in ultracold gases [51].

References

[1] T.Vojta. *Quantum Phase Transitions. In: Computational Statistical Physics.* Springer, Berlin, Heidelberg, 2002, p. 211-216.

[2] G. H. Lai. Quantum phase transitions, 2006. `https://pdfs.semanticscholar.org/2746/a49ab619e70925c0e2386e5520669c0aa560.pdf`, last accessed on 13-06-2020.

[3] E. M. Kessler et al. Dissipative phase transition in a central spin system. *Physical Review A*, **86**.1, 2012: 012116.

[4] A. O. Caldeira, A. J. Leggett. Influence of dissipation on quantum tunneling in macroscopic systems. *Physical Review Letters*, **46**, 1981, p. 211-214.

[5] A. O. Caldeira A. J. Leggett. Quantum tunnelling in a dissipative system. *Annals of physics*, **149**.2, 1983, p. 374-456.

[6] F. L. Vernon Jr. R. P. Feynman. The theory of a general quantum system interacting with a linear dissipative system. *Annals of physics*, **24**.2, 1963, p. 118-173.

[7] A. J. Leggett et al. Dynamics of the dissipative two-state system. *Reviews of Modern Physics*, **59**.1, 1987.

[8] S. Gopalakrishnan. Dissipative phase transitions. 2008. `https://guava.physics.uiuc.edu/~nigel/courses/563/Essays_2008/PDF/gopalakrishnan.pdf`, last accessed on 13-06-2020.

[9] A. Mielke T. Stauber. Equilibrium correlation functions of the spin-boson model with sub-ohmic bath. *Physics Letters A*, **305**.5, 2002, p. 275-280.

[10] A. Strathearn et al. Efficient non-markovian quantum dynamics using time-evolving matrix product operators. *Nature communications*, **9**.1 1-9, 2018.

[11] U. Weiss. *Quantum dissipative systems.* Vol. 13. World scientific, 2012, p. 40.

[12] A. Shnirman Y. Makhlin, G. Schön. Quantum-state engineering with josephson-junction devices. *Reviews of modern physics*, **73**.2, 2001: 357.

[13] I. L. Chuang M. A. Nielsen. *Quantum Computation and Quantum Information.* Cambridge University Press, Cambridge, UK, 2000.

[14] W. Hofstetter K. Le Hur, P. Doucet-Beaupré. Entanglement and criticality in quantum impurity systems. *Physical Review Letters*, **99** 126801, 2007.

[15] F. L. Vernon Jr. R. P. Feynman. Universal and measurable entanglement entropy in the spin-boson model. *Physical review letters*, **98**.22 220401, 2007.

[16] M. Vojta R. Bulla, N. H. Tong. Numerical renormalization group for bosonic systems and application to the sub-ohmic spin-boson model. *Physical review letters*, **91**.17 170601, 2003.

[17] G. Schön A. Shnirman. *Quantum Phase Transitions. In: Computational Statistical Physics.* edited by Yu. Nazarov (Kluwer, Dordrecht), 2003, p. 357-370.

[18] M. Vojta F. B. Anders, R. Bulla. Equilibrium and nonequilibrium dynamics of the sub-ohmic spin-boson model. *Physical review letters*, **98**.21 210402, 2007.

[19] H. Zheng Z. Lü. Quantum dynamics of the dissipative two-state system coupled with a sub-ohmic bath. *Physical Review B*, **75**.5 054302, 2007.

[20] C. Duan et al. Zero-temperature localization in a sub-ohmic spin-boson model investigated by an extended hierarchy equation of motion. *Physical Review B*, **95**.21 214308, 2017.

[21] Z. D. Cheng H. Wong. Density matrix renormalization group approach to the spin-boson model. *Physical Review B*, **77**.17 174305, 2008.

[22] M. Thoss H. Wang. From coherent motion to localization: Ii. dynamics of the spin-boson model with sub-ohmic spectral density at zero temperature. *Chemical Physics*, **370**.1-3, 2010, p. 78-86.

[23] A. Mielke K. S. Kehrein. On the spin-boson model with a sub-ohmic bath. *Physics Letters A*, **219**.5-6, 1996, p. 313-318.

[24] K. Le Hur. Entanglement entropy, decoherence, and quantum phase transitions of a dissipative two-level system. *Annals of Physics*, **323**.9, 2008.

[25] K. Ingersent M. Glossop. Numerical renormalization-group study of the bose-fermi kondo model. *Physical review letters*, **95**.6 067202, 2008.

[26] K. L. Wang Y. Y. Zhang, Q. H. Chen. Quantum phase transition in the sub-ohmic spin-boson model: An extended coherent-state approach. *Physical Review B*, **81**.12 121105, 2010.

[27] M. B. Plenio M. Abdi. Analog quantum simulation of extremely sub-ohmic spin-boson models. *Physical Review A*, **98**.4 040303, 2018.

[28] J. Ankerhold D. Kast. Persistence of coherent quantum dynamics at strong dissipation. *Physical review letters*, **110**.1 010402, 2013.

[29] M. Thorwart P. Nalbach. Ultraslow quantum dynamics in a sub-ohmic heat bath. *Physical review B*, **81**.5 054308, 2010.

[30] T. H. Hsieh. From d-dimensional quantum to d+ 1-dimensional classical systems. *Student review*, **2**.1, 2016.

[31] S. Sondhi et al. Continuous quantum phase transitions. *Reviews of modern physics*, **69**.1 315, 1997.

[32] R. Bulla M. Vojta, N. Tong. Quantum phase transitions in the sub-ohmic spin-boson model: Failure of the quantum-classical mapping. *Physical review letters*, **94**.7 070604, 2005.

[33] K. Ingersent S. Kirchner, Q. Si. Finite-size scaling of classical long-ranged ising chains and the criticality of dissipative quantum impurity models. *Physical review letters*, **102**.16 166405, 2009.

[34] S. Kirchner. Spin path integrals, berry phase, and the quantum phase transition in the sub-ohmic spin-boson model. *Journal of Low Temperature Physics*, **161**1-2, 2010, p. 282-298.

[35] R. Bulla M. Vojta, N. Tong. Erratum: Quantum phase transitions in the sub-ohmic spin-boson model: Failure of the quantum-classical mapping [phys. rev. lett. 94, 070604 (2005)]. *Physical Review Letters*, **102**.24 249904, 2009.

[36] M. Vojta et al. Mass-flow error in the numerical renormalization-group method and the critical behavior of the sub-ohmic spin-boson model. *Physical Review B*, **81**.7 075122, 2010.

[37] A. Winter et al. Quantum phase transition in the sub-ohmic spin-boson model: Quantum monte carlo study with a continuous imaginary time cluster algorithm. *Physical review letters*, **102**.3 030601, 2009.

[38] H. Fehske A. Alvermann. Sparse polynomial space approach to dissipative quantum systems: Application to the sub-ohmic spin-boson model. *Physical review letters*, **102**.15 150601, 2009.

[39] A. W. Chin et al. A variational description of the quantum phase transition in the sub-ohmic spin-boson model. *arXiv preprint arXiv:1102.1483*, 2011.

[40] A. W. Chin et al. Generalized polaron ansatz for the ground state of the sub-ohmic spin-boson model: An analytic theory of the localization transition. *Physical review letters*, **107**.16 160601, 2011.

[41] E. Paladino et al. Decoherence and 1/f noise in josephson qubits. *Physical review letters*, **88**.22 228304, 2002.

[42] S. W. Jung et al. Background charge fluctuation in a gaas quantum dot device. *Applied Physics Letters*, **85**.5, 2004, p. 768-770.

[43] D. Tobias et al. Origins of 1/f noise in individual semiconducting carbon nanotube field-effect transistors. *Physical Review B*, **77**.3 033407, 2008.

[44] O. Astafiev et al. Temperature square dependence of the low frequency 1/f charge noise in the josephson junction qubits. *Physical review letters*, **96**.13 137001, 2006.

[45] J. Eroms et al. Low-frequency noise in josephson junctions for superconducting qubits. *Applied Physics Letters*, **89**.12 122516, 2006.

[46] D. D. Osheroff D. Rosenberg, P. Nalbach. Memory effects in amorphous solids below 20 mk. *Physical review letters*, **90**.19 195501, 2003.

[47] K. Ingersent J. L. Smith Q. Si, S. Rabello. Locally critical quantum phase transitions in strongly correlated metals. *Nature*, **403**.6858, 2001, p. 804-808.

[48] P. Gegenwart et al. Multiple energy scales at a quantum critical point. *Science*, **315**.5814, 2007, p. 969-971.

[49] A. C. Neto C. Seoanez, F. Guinea. Dissipation due to two-level systems in nano-mechanical devices. *Europhysics Letters*, **78**.6 60002, 2007.

[50] Y. V. Nazarov G. L. Ingold. *Single Charge Tunneling.* edited by H. Grabert and M. H. Devoret (Plenum, New York), 1992.

[51] A. Recati et al. Atomic quantum dots coupled to a reservoir of a superfluid bose-einstein condensate. *Physical review letters*, **94**.4 040404, 2005.