Generating Singular Partitions

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1 Abstract

In this paper I shall outline a method to find the singular and idempotent rank and implied generating sets of a subsemigroup of the partition monoid which contains the set of all singular partitions.

2 Introduction

This paper follows the work in [1] extending the method to the partition monoid. All notation will be used consistently throughout and used consistnetly with [1]. The rank of a semigroup is defined to be the cardinality of the least generating set of the semigroup. This definition can be extended to include singular and idempotent rank of a semigroup which means that all of the generators must be singular or idempotent respectively. This definition however can only be applied to semigroups which are entirely singular or idempotent. To assign a singular or idempotent rank to a semigroup which is not comprised entirely of singular or idempotent elements the semigroup must be decomposed into the disjoint union of its singular or idempotent elements and the rest of its elements. If both of these form subsemigroups the singular or idempotent rank of the original semigroup is that of its idempotent or singular subsemigroup. For the rest of this paper I shall define the partition monoid on n points to be the set of all non empty partitions of the $2n$ elements $\{1, ..., n\}$ and $\{-1, ..., -n\}$ denoted $\mathrm{P_n}$. Composition of elements of this set will be defined in the following way. For $A, B \in \mathrm{P_n}\, AB$ can be calculated by redefining the values from $\{-1, ..., -n\}$ as values in $\{1', ..., n'\}$ for A and denoting this A' then doing the same for the values from $\{1, ..., n\}$ in B to make B'. Once this has been done AB can be realised as the connected components of a graph where the A' and B' are sets of connected components.

Let S be a subsemigroup of $\mathrm{P_n}$ such that S contains all singular partitions. This means that S can be expressed as the disjoint union between G and $\mathrm{SingP_n}$. Where G is a subset of the permutations on n points $G \subset S_n$ and $\mathrm{SingP_n}$ is the set of all singular partitions. The set $\mathrm{SingP_n}$ can be further seen to be the disjoint union of smaller singular set. $\mathrm{SingP_n} = \mathrm{SingT_n} \cup \mathrm{SingPT_n} \cup \mathrm{SingPP_n}$ where $\mathrm{SingT_n}$ is the set of partitions induced by taking transforms from $\{1, ..., n\}$

to $\{-1,...,-n\}$ and from $\{-1,...,-n\}$ to $\{1,...,n\}$. $\mathrm{SingPT_n}$ can similary be seen as the set of partitions made from partial transforms from $\{1,...,n\}$ to $\{-1,...,-n\}$ and from $\{-1,...,-n\}$ to $\{1,...,n\}$. Finally $\mathrm{SingPP_n}$ can be viewed as the partitions created from psudofunctions which map a set of multiple elements to a set of multiple images i.e $f(\{1,2\})->\{3,4\}$
By above $S=\langle G \cup \mathrm{SingT_n} \cup \mathrm{SingPT_n} \cup \mathrm{SingPP_n}\rangle$. By virtue of the permutations their partitions are all pairs of a single positive and negative value as a result they cannot by themselves be used to generate any elements $\mathrm{SingT_n}, \mathrm{SingPT_n}$ *or* $\mathrm{SingPP_n}$. Elements of $\mathrm{SingT_n}$ as a set of full rank transformations can be characterised by their entire domain having defined images even if the codomain is smaller than the domain. If you take the subset of singular transformations with $\{1,...,n\}$ as their domain and $\{-1,...,-n\}$ as their codmain the partitions that they induce can be categorised as all positive values must be in a class with one negative value and at least one class must have multiple positive values meaning that at least one negative value must be in a class by itself. $\mathrm{SingT_n}$ can be then seen as the union of the previous partitions representing transformations with $\{1,...,n\}$ and $\{-1,...,-n\}$ as their domain in turn. Next $\mathrm{SingPT_n}$ as the set of partition created by partial transformations as such their domain has to be the complete set on n points while the domain must be smaller than the n points, this is as points can have the same or no image. Taking the set $\{1,...,n\}$ to be the domain of these partial transformations the set of the partitions that this generates can be seen as the set of partitions which have at least one of their positive values in a class by itself which represents that elements image being undefined. In addition to this the remaining classes must contains at least one positive values and only one negative value or also be single positive elements or negative value, which represents that value being the image of no points. $\mathrm{SingPT_n}$ can be then seen as the union of the previous partitions with $\{1,...,n\}$ and $\{-1,...,-n\}$ as their domain in turn. Finally the partitions in $\mathrm{SingPP_n}$ can be characterised by the set of partitions which have at least one class which has at least two positive and at least two negative values in which represents part of the psudofunction, e.g $f(\{1,2\})->\{3,4\}$, the other classes of the partition can take any form. In addition to these classifications there are partitions of the form $([i,j,-k],[-i],[l,-l,-m],[-l])$ which while do not fit any of these classifications can be made from the composition of $([i,j,-k],[-i],[l,-l],[m,-m]),([i,-i],[j,-j],[l,-l,-m],[-l]) \in \mathrm{SingT_n}$ so these partitions shall be considered as part of $\mathrm{SingT_n}$ as can be formed by taking $\langle \mathrm{SingT_n}\rangle$

3 Result

The method here is based upon the one outlined in [1] with some changes outlined below in the methods section. Of note here this method is defined for a subsemigroup S of $\mathrm{P_n}$ which can be expressed as $\langle G \cup \mathrm{SingP_n}\rangle$ Where the value e is the amount of edges in the multigraph defined by taking the quotient of the complete graph on n points by G.

Theorem 3.1. *For the partition monoid on n points with $n > 2$ the singular rank of $S = G \cup \mathrm{SingP_n}$ is $2e$*

Theorem 3.2. *For the partition monoid on n points with $S = G \cup \mathrm{SingP_n}$. If the quotient multigraph of the complete graph on n point by G has two vertices and all the edges that connect them belong to the same equivalence class then the idempotent rank of S is $2e + 2$. Otherwise the idempotent rank is $2e$.*

4 Method

The method outlined here is based upon the method in [1]. First it is important to note that the equivalence classes that are generated through the quotient of the complete graphs on $2n$ points are different than that in [1]. This if for two reasons the first is that unlike most graphs of $2n$ points the labels for the vertices are $\{-n, -n+1, ..., n-1, n\}$ rather than $\{1, ..., 2n\}$ this is so the partitions as defined above can act on the graph. As elements of G are permutations as above each class that they induce pairs a single positive and negative value and every vertex of the graph must be in an equivalence class. This is not to say that the quotient graph will contain n vertices but that it will contain at most n vertices as not all permutations are transpositions and so multiple elements may be successively related.
Take the complete graph on $2n$ points as described above and from it define a new graph which vertices are the equivalence classes formed by taking the set quotient of the vertices of the complete graph by G and the same process for the edges. Denote this new graph H. Note that as S in a monoid its subsemigroup G must contain the identity partition and as such all values will be in the same equivalence class as their negative.
The edges in H can be split into two categorise this is of those which connect either two positive or two negative values and those which connect one positive and one negative value, for each pair of values $i, j \in \{-n, n\}$. Define a new subgraph H' as the subgraph formed by the latter of those categorise. Redefine H as the graph formed remaining edges not used in defining H'. There can be observed to be two types of edge in the graphs H and H'. The first are loops an edge which connect a vertex to itself. The second type of edge are the remaining edges that connect different vertices. If the graphs H and H' each have only one none loop edge this edge will be of the form $\{\{i\}, \{j\}\}$ for $i \in i, j \notin i$, this is the special case in $Theorem 3.2$ and an edge must be added between the two vertices $\{i\}, \{j\}$. These graphs must now be oriented to become a strongly connected digraph, this can be done as the the graphs are completely connected and bridgeless. From this point on e will represent the amount of edges in H. For every edge $\{\{i\}, \{j\}\}$ in H there exists the edge $\{\{-i\}, \{j\}\}$ or $\{\{i\}, \{-j\}\}$ in H', the edge in H' must be oriented in the opposite direction as the edge in H.
Each edge of these graphs will represent a partition. If the edge is the edge added for the special case this edge can be seen as the directed edge from $\{\{i\}, \{j\}\}$, let $k \in \{j\}$. This special edge represents the partition which acts like the

identity on all points which are not i, j, k or their negatives. On these points the partition is defined as $[i, -j], [j, -k, k], [-i]$, this is the generalisation of the special transformation in [1] to a partition. For the remaining edges they can be considered a directed edge from $\{\{i\}, \{j\}\}$ these edges represent the partition which acts like the identity on all edges that are not i, j or their negatives. On i and j this partition is defined as $[i, j, -j], [-i]$ also a generalisation of the ϵ in [1]. Collecting all the defined partitions into a set N and letting M be the least cardinality generating set for G then $S = \langle M \cup N \rangle$. In the case that the subgraphs has not exactly one non loop edge there are $2e$ partitions in N and in the case that there was only one non loop edge in each subgraph there are $2e + 2$ partitions in N. As such the singular or idempotent rank of S is $2e$ or $2e + 2$ for their respective cases.

5 Proof

The proof here will find an upper and a lower bound for the cardinality of the generating set and that these coincide and as such the limit if it exists must equal that value. We know that this limit must exist as the singular and idempotent rank of the singular ideal of the partition monoid is well documented.
As established in the introduction no element of $\mathrm{SingT_n}, \mathrm{SingPT_n}$ *or* $\mathrm{SingPP_n}$ can be used to generate elements of G and elements of G by themselves cannot be used to generate any of the other types elements. Looking next at $\mathrm{SingT_n}$ if this were just the partitions of transformations with $\{1, ..., n\}$ as their domain as similar argument could be applied however as this is the union of the partitions created from transformations with domain as $\{1, ..., n\}$ and $\{-1, ..., -n\}$ it is possible to generate elements of $\mathrm{SingPT_n}$. For example in the case of $n = 2$ the partition $\{[1], [2, -1, -2]\}$ is the partition generated by the transformation $f(x)$ with domain of $\{-1, -2\}$ with $f(x)$ defined in the following way

$$f(x) = \begin{cases} 1 & \text{if } x = 2 \\ x & \text{otherwise} \end{cases}$$

Similarly the partition $\{[1, 2, -1], [-2]\}$ is generated by the transformation $g(x)$ with domain of $\{1, 2\}$ defined by

$$g(x) = \begin{cases} 1 & \text{if } x = 2 \\ x & \text{otherwise} \end{cases}$$

The composition of $\{[1], [2, -1, -2]\}$ and $\{[1, 2, -1], [-2]\}$ is $\{[1, -1], [2], [-2]\}$ which fits the description for an elements of $\mathrm{SingPT_n}$ so elements of $\mathrm{SingT_n}$ can be used in the generation of $\mathrm{SingPT_n}$. Elements of $\mathrm{SingPT_n}$ cannot be used to generate elements of $\mathrm{SingT_n}$ as a positive or negative value is not connect to the set $\{1', ..., n'\}$ which is defined in the binary operation cannot be connected it to any other values. This precludes $\mathrm{SingPT_n}$ from generating elements of $\mathrm{SingT_n}$ as for the partitions of transforms every value from its domain must

be connected to an element in its range this would force the partial transforms to be defined with their images the domains of the transforms but this would then mean that they map one point to many images then fail the definition of a function. The same argument justifies why elements of $\mathrm{SingPP_n}$ cannot be used in the generation of $\mathrm{SingT_n}$. Partitions in $\mathrm{SingT_n}$ can be used in creation of $\mathrm{SingPP_n}$ as it is possible to generate the sets of multiple positive values connect to negatives by taking the transformation with the positive values defined as its domain which maps them all to the value $1'$ defined in the binary operation. This transformation then need to be composed with the transformation with domain of the negative values which maps the desired negative values also to $1'$, the remaining points in both transformations can be either mapped in the same way to each other made to be undefined as shown in the generation of $\mathrm{SingPT_n}$ or mapped as in transforms.
This argument is useful because it shows that a minimal cardinalilty generating set for $\mathrm{SingT_n}$ can be used to generate the remaining elements.
Throughout the following proofs I will be choosing partitions which have the highest amounts of connected sets of components this is as the through composition on transformations with a equal domain you cannot increase the number of connected sets of components. This is means that for a minimal generating set the partitions such as these should be chosen of maximal sets of connected components as these components can only be made from initial inclusion or composition of other elements of maximal sets of connected components. It is also important to note transformations with the domain of $\{1, ..., n\}$ cannot be used to generate transformations with domain of $\{-1, ..., -n\}$ this can be seen that the transformation monoid is closed and contains all of the transformation with domain of $\{1, ..., n\}$ and none with domain of $\{-1, ..., -n\}$.

5.1 Upper Bound

The method outlines in [1] outlines that the cardinality of the least generating set for a set of transforms is the amount of edges in the quotient of the complete graph on n points by the permutation set G. For these graphs there are e or $e + 1$ edges. The value of e from [1] are the same as the ones defined in method section here. This is as the identity partition will be in G and taking quotient by the identity will reduce the graph to having n nodes as each value is equivalent to its negative. As transformation are only defined on positive values it suffices to look at the edges of graph H. The edges in H can be seen as the equivalence class of edges which only connect positively valued nodes quotient by the graph action of G, this is as edges in H both have positive values of nodes. This representation of edges of H is the same as taking the quotient of the complete graph on n points by G this is how the edges in [1] are defined as such the values of e are equal. The transformations defined by these e or $e + 1$ edges are a minimal singular or idempotent generating set for the transformations on n points by [1]. This means that the $2e$ or $2e + 2$ edges of H and H' represent the minimal singular or idempotent generating set for the partitions which transformations with $\{1, ..., n\}$ and $\{-1, ..., -n\}$ as their domains represent. By the logic above this set of partitions that represent the transformations can be used to generate all of $\mathrm{SingP_n}$. As such the singular or idempotent rank of is at most $2e$ or $2e+2$ in the case where the quotient graph has only two vertices and one none loop edge.

5.2 Lower Bound

For the lower bound the special case shall be considered separately from the general case. Partitions with n sets of connected components are partitions generated by elements of G therefore the maximum number of connected components for elements of the minimal generating set will have is $n-1$. By composition under the binary relation the number of connected components cannot increase only decrease by becoming connected by the relations. As such a minimal generating set must comprise of partitions with $n-1$ connected components and be able to generate all other partitions with $n-1$ connected components.

5.2.1 General Case

For this case we will assume that the $2e-1$ partitions that the edges of the graphs H and H' represent except for the partition represent by $\{\{i\},\{j\}\}$, I will show that you cannot make the partition which pairs all values which are not i, j to their negatives then $[i, j, -j], [-i]$. Let this partition be referred to as J. For J to be obtained all of the elements which would make it must also have rank of $n-1$ and also have the $-i$ as the value which is not connected to anything other value. This is as if there were more than $-i$ not connected to any other value their number of connected components would be too few and and if $-i$ were connected to any other negative values it would not be able to be disconnected from them and still retain a partition with $n-1$ connected components. $-i$ can not be connected to positive values but this must be shown by two cases. $-i$ cannot be connected to three positive values as this would leave to few elements to make the partition have $n-1$ connected components. $-i$ if connected to two positive values to have a final product which has $-i$ not in a class with any other value the partitions that you multiply it by would have to have the two positive values each in class with no other value. If that were the case the partition would have a less that $n-1$ connected components and such then not be able to be used to make J as J has $n-1$ connected components. If $-i$ is connected to one positive value it can either be connect to i or a value which isn't i, in the case where it is connected to i it behaves like the identity on i and such cannot make J. When $-i$ is connected to a positive value, k, which isn't i then at least one other of the elements used to make J must have $-k$ in a class by itself. This means that elements must have a block in it which contains three elements, when composed with the partition with $-i$ is connected to one positive value this would mean that the product would either have one block with four connected elements or two blocks with 3 connected elements, as such its rank would be below $n-1$ and cannot produce J. If $-i$ is connected to no positive values it would mean that J is obtainable by the application of permutations. This would mean that elements of G would be able to be applied to the partition and obtain J this means that the edge would be in the same edge equivalence class as the edge $\{\{i\},\{j\}\}$ and so would be against the way the $2e-1$ permutation were picked and so cannot be a case. Therefore J cannot be made from other partition of rank n from G or other partitions of rank $n-1$ therefore must be included in the minimal generating set. As such the minimal generating set must be of size $2e$ at least.

5.2.2 Special Case

This case is for when the quotient subgraph H has two vertices and one none loop edge. Let the two vertices be the equivalence classes $\{i\}$ and $\{j\}$ and the graph have the none loop edge directed as $\{\{i\},\{j\}\}$. This graph will also contain any number of loops. I will show that the $2e$ edges of the graph H and H' cannot generate the partitions which behave like the identity on all values which are not hi, j, k and $[i,-j],[j,-k,k],[-i]$ on h,i,j,k when $h,i \in \{i\}, h,i \notin \{j\}, j,k \in \{j\}$. Let the previously described specific transformation be called J. J has an equivalent from H' , $[j,-i],[i,-h,h],[-j]$ call this J'. J has $n-1$ sets of connected components and as such must be made from composition of partitions with $n-1$ or n sets of connected components, thus it suffices to show the partitions from the edges of H, H' and G cannot make J. In J the values $i,-j$ are connected in $[i,-j]$. Partitions from G do not connect values from different equivalences classes as they are what defined the graphs equivalences classes as such can not connected i,j at any point. Considering the loops of H and H' they are partitions of $n-1$ connected components. For loops on $\{i\}$ they behave like the identity on all values in $\{j\}$ and the equivalent holds true for loops on $\{j\}$ as such loops never connect any values from $\{i\}$ to $\{j\}$. The one none loop edge of each H and H' are the only edges which connect a value from $\{i\}$ to $\{j\}$ and these two partitions are of the form, $[j,i,-i],[-j]$ and $[-i,-j,j,[i]$ with all other values behaving like the identity from H and H' respectively. In J, i,j are connected to only one negative value this means that i,j must be connected or elements from their classes must be connected in the elements to produce J. By the previous arguments on elements of G and loops they connect no elements of $\{i\},\{j\}$ so only the case where i,j are connected must be considered in making J. So the partitions for the none loop edges of H and H' are the only partitions which connect values of $\{i\},\{j\}$ and therefore must be used in the construction of J and J'. Considering J i,j are in different blocks and so must be separated from the blocks they are in, in the none loop edge partitions. To do this i,j must be assigned to different blocks of size one then re connected by later multiplication however to assign them to different blocks of size one this requires a partition with $n-2$ connected components. As no partitions with $n-2$ connected components are in this generating set J cannot be generated nor can J' so the size of the minimal generating set is at least $2e+2$.

References

[1] K. A. Kearnes, Á. Szendrei, and J. Wood. Generating singular transformations. *Semigroup Forum*, 63(3):441–448, 2001.