Generating Partial Transformation and Partition Monoids

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Aims

This project aimed to find the singular and idempotent rank for partial transformation and partition monoids. The scope of this result only applies to monoids which contain all singular and idempotent elements in addition to other elements. This is to say find the fewest number of singular or idempotent elements which could be used to generate one of these monoids.

Uses

The use of finding these ranks is two fold, the first is the classification of more complex mathematical objects in terms of simpler ones. The second is in finding these ranks a method is defined which can be used to return the generating set for one such monoid or used in reverse to quickly find the singular or idempotent closure of a subset of the symmetric group. Having methods that can quickly return generating sets and closures is helpful in many computational applications of mathematics as it means processes can be ran much faster and use less memory.

Glossary

- Monoid: A semigroup with an identity.
- Semigroup: A set with a binary operation which is closed and associative.
- Partition: A collection of the values –n to n which are grouped into blocks of connected elements.
- Partial Transformation: A function which can have undefined mapping for given values.
- Symmetric group: The group of permutations on n points.
- Idempotent: A property of a function which means that it is constant when applied to itself.
- Singular: A property of a function that means its rank is smaller than its degree, for this context it means the range of a function is smaller than its domain
- Graph: A collection of vertices and edges, the vertices can be seen as points and the edges lines connecting them

Method

The method the results for partition and partial transformation monoids rest on is an elevation of the method in Kearnes, [1]. The method will be outlined here with an example in the following section. This method relies on taking the monoid in question, S, and splitting it into a disjoint union of is non singular elements, G, and its singular elements, SingP. Letting n be the degree of the monoid, the complete graph of n points is then has its quotient taken by G to form a new graph. The methods for partial transformations and partitions now become separate.

Partial transformations:

For partial transformations the graph must be oriented to become strongly connected. The oriented edges now become transformations which are constant on that edge and act as the identity on all other points. From each vertex of the graph an edge must be added to a new vertex which shall be denoted '-'. For the edges which connect a vertex to '-' these represent the partial transformation which is defined as the identity on all values except one representative from the equivalence vertex class, on that value it is undefined. The collection of partial transformations which these edges represent along with the generators for G are the desired generating set.

Partitions :

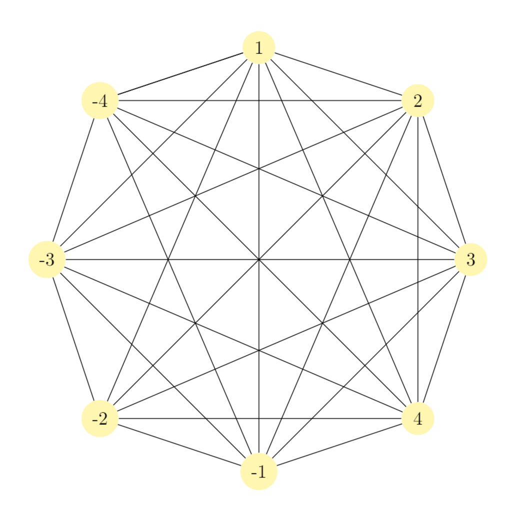
In the partitions method it is important to note that the complete graph which you quotient is the complete graph on 2n points with labels –n to n. This is as an n degree partition contains the values –n to n. Taking the quotient of this graph is done in the same way however once the quotient graph has been calculated it must be divided into subgraphs. These subgraphs are of the edges which connect values of the same sign and those which connect values of different signs for a given pair of vertices. These subgraphs must then be oriented in opposing direction to create strongly connected graphs. Each edge can now be taken to represent a partition which connected the start and end values to the negative end value and connected all other values with their negative. The partitions that the edges represent are the partitions that form the idempotent or singular elements in the generating set.

Both of these methods can also deal with the case where the quotient graph has only two vertices and a single none loop edge; as while this cannot be oriented to become strongly connected ,as they are not bridgeless, an additional edge can be added to resolve this.

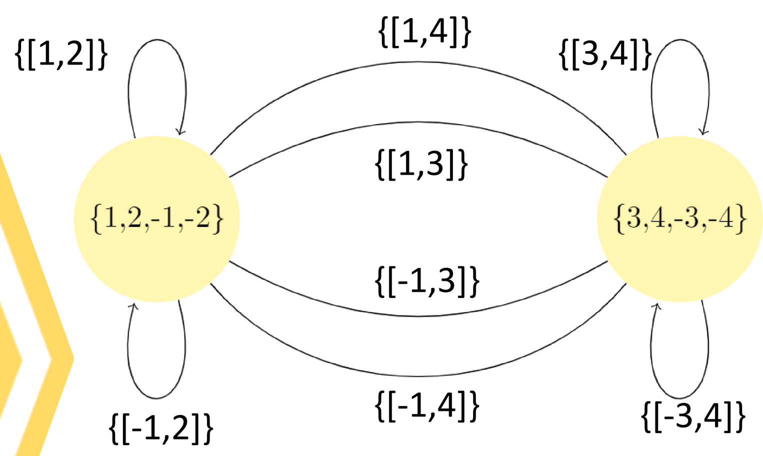
While the proof has been omitted from this poster, the proof works by considering the lower and upper bound for the ranks. These bounds are then shown to coincide and are well known to exist so take the common value.

Example

The example case that shall be worked through is applying the method to a partition monoid. The partition monoid which shall be called S will contain the partition representation of the permutation subgroup < (1,2) , (3,4) > , G, and all singular partitions on 4 points, SingP. S can be seen as < G U SingP > so the method applies to this case. As this example works on 4 points the graph that can be seen to used in the first step is the complete graph on 8 points with labels -4 to 4. The graph being complete means that every vertex is connected by a single undirected edge to every other vertex. The next graph is the quotient of this complete graph by G, this causes the vertices and edges to become equivalence classes. This graph is then separated and oriented accordingly. The Final panel displays a visual representation of the idempotent partitions first followed by the partition representation of the permutation subgroup < (1,2) , (3,4) > .



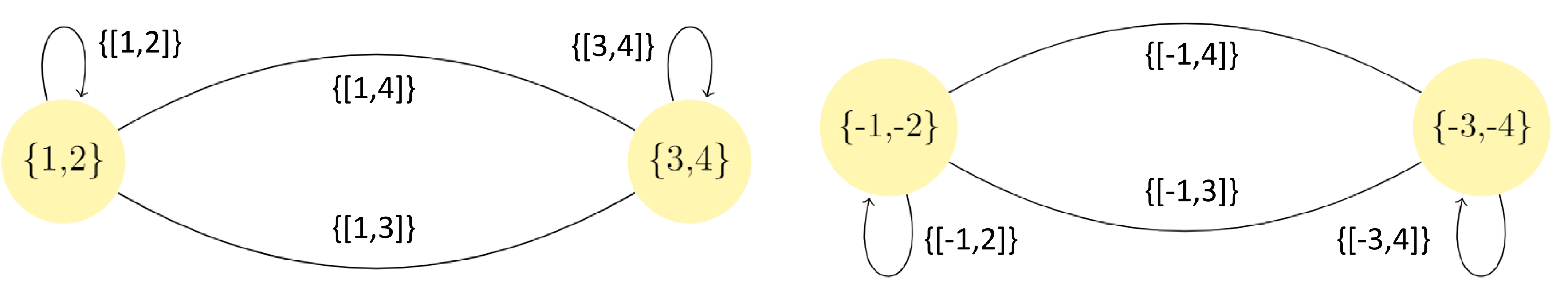
Complete graph on 8 points.



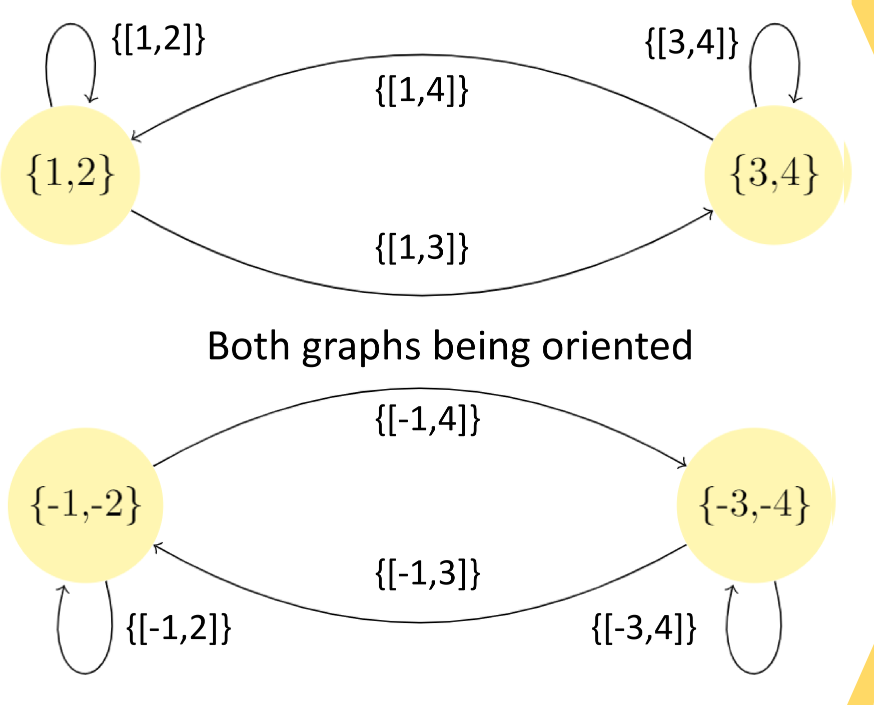
Quotient graph of the complete graph on 2n points by G giving edges and vertices as equivalence classes.

Acknowledgements

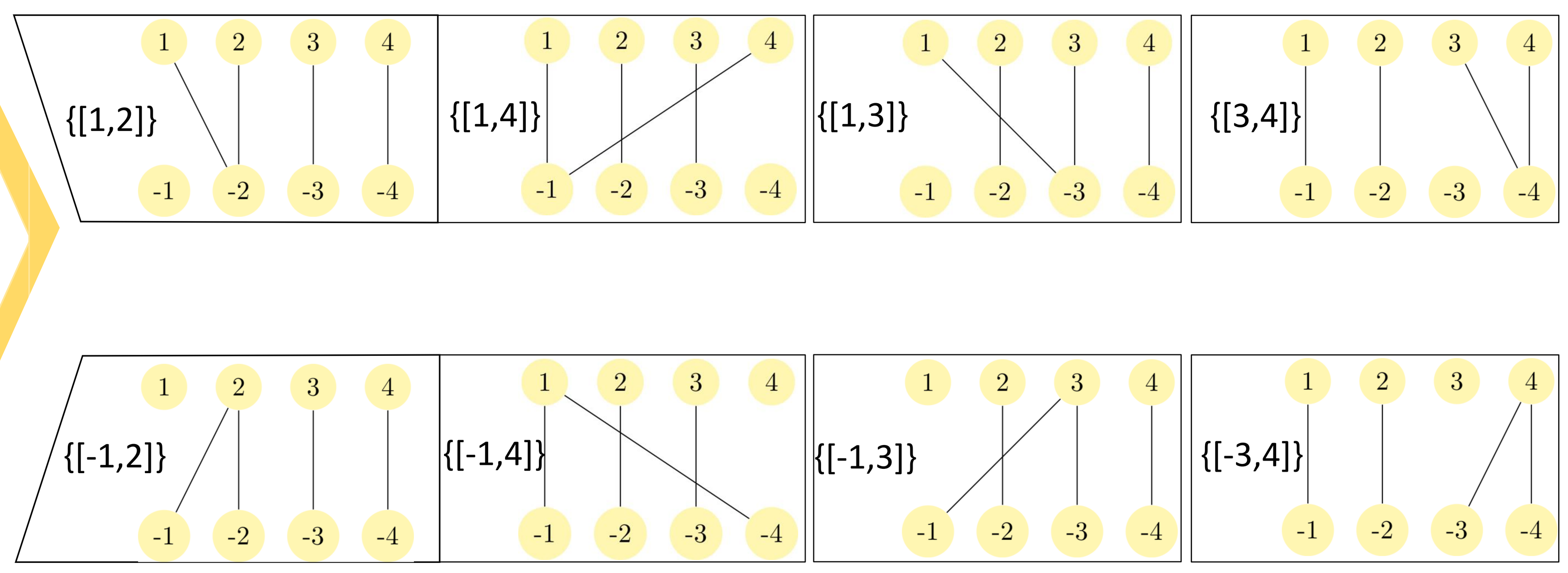
I would like to thank my supervisor got the support they have given me through this project and Lord Laidlaw for this opportunity.

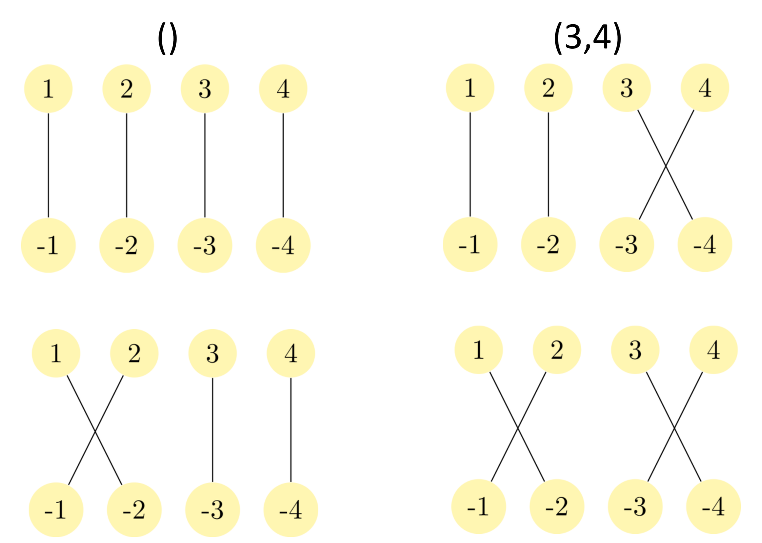


Two subgraphs taken from the quotient graph



Both graphs being oriented





Results

In addition to the methods discussed above and the proof which under pin it the methods can be generalised to provide expressions for the singular and idempotent rank of certain monoids. For the following results let e be the number of edge in the quotient of the complete graph by G and v be the number of vertices in the same graph. Let all monoids be of the form of S = G U SingP.

For the partition monoid on n points with n > 2 the singular rank is 2e similarly if the partial transformation monoid on n points with n > 2 the singular rank is e + v.

The next results are for if the quotient multigraph of the complete graph on n point by G has two vertices and all the edges that connect them belong to the same equivalence class then the partition monoid has idempotent rank of 2e+2 while the partial transformation monoid has idempotent rank e+1+v. Otherwise then the idempotent rank is the same as the singular rank

Reference

[1] K. A. Kearnes,́ A. Szendrei, and J. Wood. Generating singular transformations.Semigroup Forum, 63(3):441–448, 2001.