An Investigation into the Rado Graph and Complex

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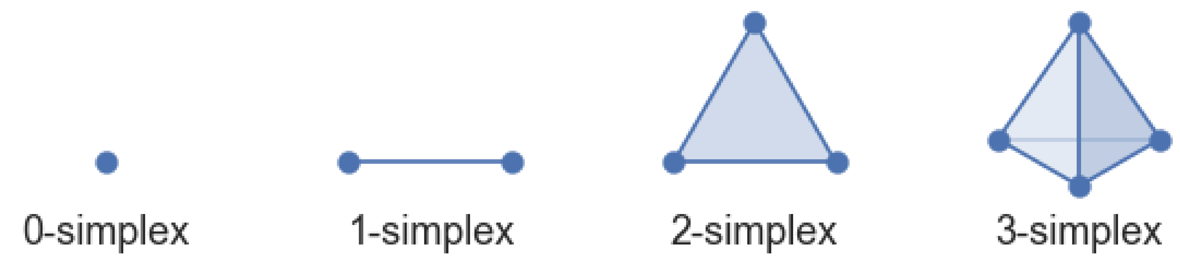
Introduction

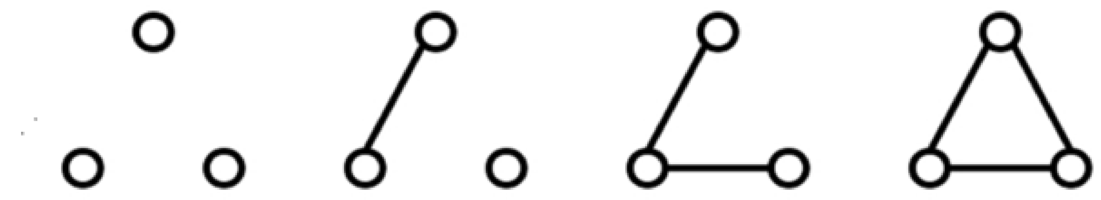
My work has focused on the intersection between two seemingly distinct combinatorial structures: simplicial complexes and the Rado graph. Simplicial complexes can be understood as generalisation of graphs, where instead of simply looking at edges between points, we can look at triangles between 3 points, tetrahedrons between 4 points, and so on into higher dimensions.



[1]

These simplexes can be thought of as the first building blocks of simplicial complexes, so a graph is just a simplicial complex with only 0 and 1 dimensional simplexes, because it is made only from points and lines.

The second structure is a graph known as the Rado graph, which is best introduced through the following example. Take any finite number of points, say three, and between any two points flip a coin, and draw a line if it comes up heads. This process could produce any of the graphs below, which are all different no matter how they are labelled:



[2]

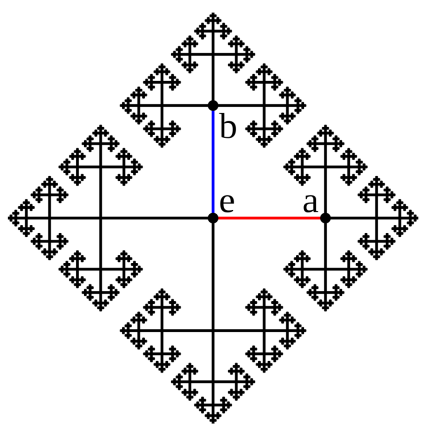
However, something very strange happens when we repeat the process with a countably infinite number of points: with probability 1, we will form the Rado Graph! This graph has some very strange properties, and a rich group of symmetries.

The intersection of these ideas leads to the Rado complex [3], a simplicial complex which shares the defining features of the Rado graph (universality and homogeneity). The question I desired to answer was a simple one:

What symmetries (if any) exist for the Rado complex? What can this tell us about the Rado graph?

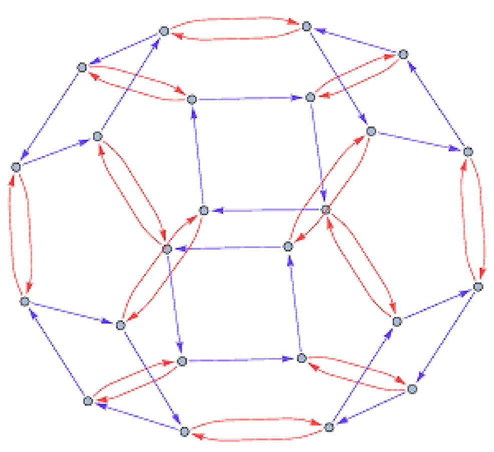
Line of Inquiry

In order to find symmetries of the Rado complex, I started with looking at specific groups of the "nicest" symmetries of the Rado Graph, which are regular automorphism groups. The key property of regular permutation groups of graphs that makes them easy to work with is that they can be represented as Cayley graphs. This is because they are determined entirely by the "connection set" of a single point, as shown to the right. I extended this idea to one of a Cayley complex, which is a simplicial complex that possesses this same key idea: it is determined entirely by its connection set. Hence, the entirety of the complex is defined by the behaviour of the complex at a single point, such as the identity.



[4]

Examples of infinite Cayley graphs. Notice how each graph looks the "same" at each point, as in each point is joined to 4 others in the above example, and 3 others below.

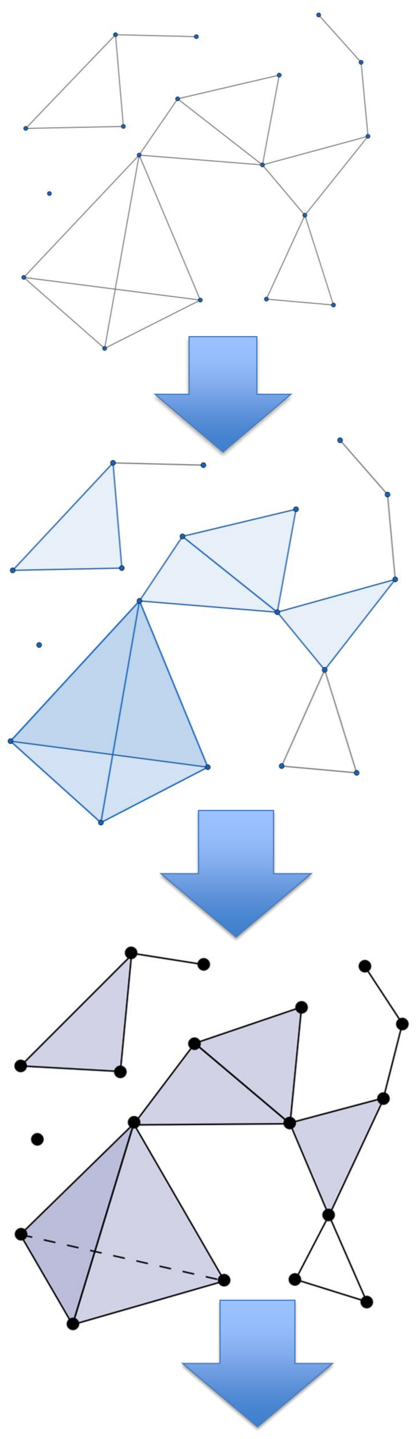


[5]

So, I used these Cayley graphs to build Cayley complexes up "layer by layer", ensuring that at each stage the resulting complex has an extra dimension, whilst still admitting the initial automorphisms and retaining the defining properties of a Rado complex. This process is sketched out opposite.

Technically this is done non-constructively, showing that a "random" complex that admits the original group G must be an n-dimensional Rado complex with probability 1, so there must be at least one n-dimensional Rado complex that admits G.

Taking this process to the limit, we can create a simplicial complex that is isomorphic to the Rado complex, whilst admitting the initial automorphisms.



[6]

Results and Summary:

I initially employed the above approach with a simple group of regular automorphisms, the cyclic group, until I was able to extend my method to more and more general groups of regular automorphisms. I was eventually able to reach the following conclusion**: The group of symmetries of the Rado complex, despite being an object with much more structure than the Rado graph, has many more similarities to the group of symmetries of the Rado complex than one might have imagined. In particular*, every regular group of automorphisms of the Rado graph corresponds to a regular group of automorphisms for the Rado complex*.** One may be able to extend the methods used here in several ways, such as giving the cardinality of the automorphism group of the Rado complex and some conjugacy classes of the Rado graph's automorphism group. Aside from this, interesting next steps would be to try to find symmetries of the Rado graph which are not symmetries for the complex, or else to show that the automorphism groups are in fact isomorphic. However this question is answered, my work has at least shown that there is a question to answer!

I would like to thank Lord Laidlaw for providing me with the opportunity to carry out this work, as well as to my supervisor Peter Cameron, and to the whole Laidlaw team for their help over these stressful few months.

References:
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