An Investigation into the Rado Graph and Complex

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1 Introduction

My work has focused on the intersection between two seemingly distinct combinatorial structures: simplicial complexes and the Rado graph.

Simplicial complexes can be understood as generalisation of graphs, where instead of simply looking at edges between points, we can look at triangles between 3 points, tetrahedrons between 4 points, and so on into higher dimensions. These simplexes can be thought of as the first building blocks of simplicial complexes, so a graph is just a simplicial complex with only 0 and 1 dimensional simplexes, because it is made only from points and lines.

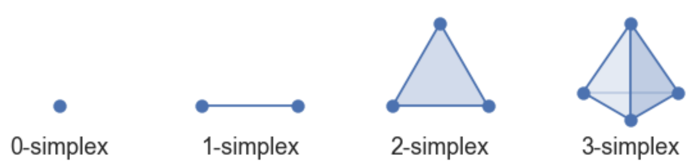

Figure 1: The first four simplexes [6]

The second structure is a graph known as the Rado graph, which is best introduced through the following example. Take any finite number of points, say three, and between any two points flip a coin, and draw a line if it comes up heads. This process could produce any of the graphs below, which are all different no matter how they are labelled.

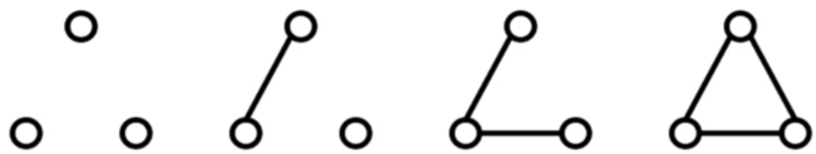

Figure 2: All graphs (up to isomorphism) that can be randomly generated from 3 points with non-zero probability[7]

However, something very strange happens when we repeat the process with a countably infinite number of points: with probability 1, we will form the Rado graph! The defining properties of this graph are universality and homogeneity: every countable graph can be found as an induced subgraph of the Rado graph; and every isomorphism between two induced subgraphs can be extended to an automorphism of the whole graph.

In a paper due to Farber, Mead and Strass[4], they create a simplicial complex which shares these defining properties (universality and homogeneity) in the context of simplicial complexes. Not only this, but if you look at the 1-skeleton of this object, which is the simplicial complex with all faces with more than 3 points removed, you once again get the Rado graph. Furthermore, although it is less straightforward than in the graph context, there is a way of "randomly generating" simplicial complexes that gives, with probability 1, the Rado complex! They called this object the Rado complex, since in many ways it appears to be a generalisation of the Rado graph.

A natural question when faced with this new object is what symmetries it has (if any besides the trivial automorphism), which would be interesting in its own right, but may also shine further light on the Rado graph itself.

My research has been focused on looking at the regular subgroups of automorphisms of this object. A regular permutation group is a permutation group such that for any pair of points in the graph, there exists exactly one permutation from one of these points to the next. Automorphisms are a special case of permutations that are not only permutations of points, but are also isomorphisms (so preserve edges, triangles etc.).

In fact, I was able to classify every regular group of automorphisms of the Rado complex through the following theorem:

Theorem 1.1. *A countable group G is isomorphic to a subgroup of Aut(R) that is regular on the vertices of R if and only if G is isomorphic to a subgroup of Aut(Δ) that acts regularly on the vertices of Δ, where R denotes the Rado graph and Δ denotes the Rado complex.*

2 Preliminaries and Definitions

Firstly, I will give a definition of any domain specific terms I will use in this paper, and seek to give an intuition for any unfamiliar ones.

2.1 Simplicial Complexes

In a more formal sense, a simplicial complex X is an ordered pair (V(X), F(X)), where V(X) is a set of points, and F(X) is a subset of the power set of V(X) that satisfies the following conditions:

Condition 2.1. *For every v in V(X), v is in F(X);*

Condition 2.2. *For every set in F(X), all subsets of that set must also be in F(X).*

2.2 Graph Theory

The Rado graph can be characterised by the following property[3]:

Definition 2.3. *Given any set of points* $\{u_1, ..., u_n, v_1, .., v_m\}$*, we can find some point w such that* $\{u_i, w\}$ *is an edge for all i, but* $\{v_j, w\}$ *is not an edge for all j. Any two graphs with this property must be isomorphic.*

We also give the definition of a Cayley graph:

Definition 2.4. *A Cayley graph, denoted Cay(G,S), is a graph with point set G such that {x,y} is an edge if and only if* xy^{-1} *is in S, and S is closed under inverses and doesn't contain the identity.*

We present the following theorem, which shows that when looking at Regular permutation groups, the natural objects are Cayley graphs:

Theorem 2.5. *A graph* Γ *is a Cayley graph if and only if there exists a subgroup* $R \leq Aut(\Gamma)$ *with R regular on vertices. In this case* $\Gamma \cong Cay(R, S)$ *for some S. [5]*

We will then generalise this notion of a Cayley graph to simplicial complexes by looking at an object I've termed a "Cayley complex":

2.3 Cayley Complex

Definition 2.6. *A Cayley complex of a group G, denoted as* $Cay(G, S_1, S_2, ...)$*, is a simplicial complex, where an n-dimensional set* $\{g_1, ..., g_{n+1}\}$ *is a face of the complex if and only if there exists* $g \in G$ *such that* $\{g_1, ..., g_{n+1}\}g = \{e, h_1, ..., h_n\}$ *and* $\{h_1, ..., h_n\} \in S_n$*. Each of the sets* S_n *must satisfy the following:*

For any $X = \{g_1, ..., g_{n+1}\} \in S_n$, we must have that all proper subsets of X are in S_i for some $i < n$.

This ensures that a Cayley complex is a simplicial complex.

For any S_n, we have that it is "closed under inverses", in the sense that it has the following property:

$$\{g_1, ..., g_n\} \in S_n \iff \{g_1^{-1}, g_2 g_1^{-1}, ..., g_{n+1} g_1^{-1}\} \in S_n$$

This is included to ensure that all translations of a face must be included in the Cayley Complex, as there are multiple ways that a face could be translated to have a vertex as the identity.

For any $\{g_1, ..., g_n\} \in S_n$ such that $g_i \neq g_j \ \forall i, j$, we also require $g_i \neq e \ \forall i$.

This ensures that we aren't defining a face where two points are the same, which is the natural extension of the connection set of a Cayley graph not containing the identity.

Define a Cayley n-complex to be a Cayley complex but with faces of at most n-dimensions. We will denote this by $Cay(G, S_1, ..., S_n)$

The crucial aspect of the Cayley graph that has been generalised here is how it is entirely determined by the connection set of the identity element, as all other edges are simply translations of these elements.

3 Proof of Results

3.1 Cayley Complexes and Regular Automorphisms

Initially, I looked at a simple kind of automorphism, cyclic automorphisms. I was able to show that cyclic automorphism groups acted on every n-skeleton, but I was unable to find an action that worked for every n-skeleton. I had a lot of trouble doing this, and tried several approaches without success.

I then decided to change my approach, and instead of seeking to find a single action that worked for all n-skeletons, I constructed a simplicial complex layer by layer from these n-skeletons. This was a subtle shift in strategy, and frustrating because in hindsight it appeared to be such a simple decision, but it greatly impacted my rate of progress.

It was this shift in focus that first lead me to define a Cayley complex, and the following theorem shows why they are incredibly useful to investigate:

Theorem 3.1. *Let G be a group, and $K = Cay(G, S_1, S_2, ...)$ be a Cayley Complex. Then $G \cong H \leq Aut(K)$, where H is the permutation group corresponding to right multiplication by elements of G, and is regular on the vertices of K.*

Proof. Let H be the permutation group corresponding to right multiplication by elements of G. Then, by Cayley's Theorem, G ≅ H.

This subgroup of Sym(K) is regular, since for any $g, h \in G$, there exists precisely one element, $g^{-1}h$, such that $g(g^{-1}h) = h$. Now, to show that $H \leq Aut(K)$, let $g_1, g_2, ..., g_n \in G$ be distinct. Then $\{g_1, g_2, ..., g_n\} \in K \iff \exists h \in G : \{g_1, ..., g_n\}h = \{e, h_1, h_2, ..., h_{n-1}\}$, where $\{h_1, h_2, ..., h_{n-1}\} \in S_{n-1} \iff \exists h \in G : \{g_1 g, ..., g_n g\}g^{-1}h = \{e, h_1, h_2, ..., h_{n-1}\}$, where $\{h_1, h_2, ..., h_{n-1}\} \in S_{n-1}, g \in G \iff \{g_1 g, ...g_n g\} \in K \iff \{g_1, ...g_n\}g \in K$, and hence every element of H is an automorphism of K, and hence we are done. □

3.2 Regular Automorphisms of the Rado Complex

Theorem 3.1 allows us to extend the method used by Cameron [1] into Cayley complexes, as it shows that simply constructing a Cayley complex which is isomorphic to the Rado complex will give us the desired result for a given regular automorphism group.

Although I began with the group of cyclic automorphisms, I was able to generalise the main line of proof repeatedly, until the conditions matched those of the more symmetric Rado graph.

These conditions are given here, and were demonstrated by Cameron to be sufficient to characterise all regular automorphism groups of the Rado graph [2]

Condition (†). *Let G be a countable group such that G is not a union of translates of square-root sets of the form* $(\sqrt{uv^{-1}})v$ *for* $u \in U$ *and* $v \in V$*, with* $U \cup V$*, where* U, V *are finite.*

It follows that we must simply prove the following to prove Theorem 1.1:

Theorem 3.2. *A countable group G is isomorphic to a subgroup of Aut(Δ) that is regular on the vertices of Δ if and only if G satisfies (†)*

⇒: Suppose that G is isomorphic to a subgroup, H, of Aut(Δ) that is regular on the vertices of Δ. Then since the 1-skeleton of Δ is isomorphic to the Rado graph, R, we have that H is isomorphic to a regular subgroup of Aut(R), and thus it must satisfy (†) by [2].
⇐: Suppose that G satisfies (†). We will then rely heavily on the following lemma, and its method of proof used in the paper *"An investigation of countable B-groups"* [1]:

Lemma 3.3. *Suppose that the countable group G cannot be expressed as the union of finitely many translates of non-principal square-root sets and a finite set. Then almost all Cayley graphs for G are isomorphic to R.*

We will show that almost all Cayley n-complexes are isomorphic to the n-skeleton of the Rado complex, and then infer that almost all Cayley complexes must be isomorphic to the Rado complex.

Firstly, from the proof in [1], there exists some S_1 such that $Cay(G, S_1)$ is isomorphic to the Rado graph, and for any set $X = \{u_1, u_2, ..., u_n, v_1, v_2, ..., v_m\}$, we can construct a sequence $(w_n)_n$ such that

$$x * (w_n)^{-1} \neq x' * (w_m)^{-1}, (w_m) * x^{'-1} \ \ \forall x, x' \in X.$$

Then, let $S_2 \subseteq \{\{g, h\}|g, h \in G\}$ be chosen randomly such that it satisfies the properties listed in section 2 necessary for $Cay(G, S_1, S_2)$ to be a Cayley complex.

Then we look at $Cay(G, S_1, S_2)$, which we will label R_2 for convenience. We will show that if G satisfies (†), then almost all Cayley 2-complexes of G using S_1 are isomorphic to the 2-skeleton of the Rado complex.

3.3 2-Skeleton of the Rado Complex

Since the 2-skeleton is the Fraisse limit of the class of all finite 2-dimensional simplicial complexes, which follows trivially from the definition of a Fraisse limit [3], we just need to show that R_2 satisfies the following condition:

Condition 3.4. *If A and B are finite 2-dimensional simplicial complexes with $A \subseteq B$ and $|B| = |A| + 1$, then every embedding of A into R_2 can be extended to an embedding of B into R_2.*

Let A' be an embedding of A, where the point set of A' is $\{u_1, ..., u_n, v_1, ..., v_m\}$, denoted by X. We want to construct a sequence $(w_n)_n$ such that the events $\{x_1, x_2, w_i\}$, $\{x_3, x_4, w_j\} \in R_2$ are independent, where $x_1, x_2, x_3, x_4 \in X$ and $\{x_1, x_2, w_i\} \neq \{x_3, x_4, w_j\}$

Now, they are independent if there exists no g in G such that $\{x_1, x_2, w_i\}$ $= \{x_3, x_4, w_j\} * g$

Without loss of generality, there are two situations in which this can occur:

$$x_1 = x_3g,\ x_2 = x_4g,\ w_i = w_jg \qquad (*)$$

$$x_1 = x_3g,\ x_2 = w_jg,\ w_i = x_4g \qquad (**)$$

Now, let the sequence $(w_n^{'})_n$ be as constructed in Theorem 3.1 of Cameron's paper[1]. Then look further at case (**).

This is equivalent to $x_3^{-1}x_1 = w_j^{-1}x_2 = x_4^{-1}w_i = g$, which implies $x_4w_j^{-1} = w_ix_2^{-1}$, which cannot occur if our sequence $(w_n)_n$ is a subsequence of $(w_n^{'})_n$

Next, we will show that we can construct a subsequence $(w'_{n_k})_k$, such that (*) holds for no element of the subsequence.

Assume we have $w_{i_1}^{'}w_{i_2}^{'}...w_{i_N}^{'}$ that do not satisfy (*). Then we want to pick $M > i_n$ such that

$$(w_{i_j}^{'})^{-1}w_M^{'},\ (w_M^{'})^{-1}(w_{i_j}^{'}) \neq x_1^{-1}x_2\ \forall j = 1, 2, ..., N;\ x_1, x_2 \in X$$

Now, $|X| = n + m$, so there are at most $(n + m)^2$ values of the form $x_1^{-1}x_2$.

Since ay=z and by=z implies a = b, there are therefore $2N(n+m)^2$ distinct values that do satisfy (*), and since the elements of $(w_n^{'})_n$ are all distinct, we have that we must be able to choose some M with the desired property.

Let $i_{N+1} = M$

Hence we can construct a subsequence of $(w_n^{'})_n$ such that no elements of the sequence satisfy (*) or (**), which we will denote $(w_n)_n$. **This sequence has the property that any two sets of the form $\{x_1, x_2, w_i\}$, where $x_1, x_2 \in X$, have the same probability of being in R_2, and are independent events.**

Then, for each l, look at the events:

$$\{u_i, w_l\} \in R_2\ \forall i = 1, 2, ..n;\ \{v_i, w_l\} \notin R_2\ \forall i = 1, 2, ...m;\ \{x_i, x_j, w_l\} \notin R_2\ \forall x_i, x_j \in X$$

Each event has probability 1/2 of happening, and they are independent as shown earlier, so the probability that all of these events doesn't happen can be denoted by some q such that $0 < q < 1$. Hence, the probability that all of these events don't happen for any w_n is the limit $(1-q)^K$ as K tends to ∞, which is 0, and thus with probability 1 we have that there exists some w_l which satisfies this property, so A' can be extended to an embedding of B' with probability 1. Note here that we assume that B is just a copy of A with one additional point and no extra faces. Although this isn't the general case, the argument works precisely the same, it's just a bit more awkward to state the events like we have done above.

There are countably many choices for A and B, and thus because a countable intersection of almost certain events is almost certain, we have that almost all Cayley 2-complexes based on S_1 are isomorphic to the 2-skeleton of the Rado complex, and thus there exists some Cayley 2-complex which is isomorphic to the 2-skeleton of the Rado complex.

3.4 2-Skeleton to the Entire Complex

We can repeat this process inductively, giving a sequence $(S_n)_n$ such that for any n, $Cay(G, S_1, ..., S_n)$ is isomorphic to the n-skeleton of the Rado complex. Use this sequence to form a Cayley complex $Cay(G, S_1, ...)$ which must then be isomorphic to the Rado complex, finishing the proof. $\square$

To justify the above claim, note that ampleness is the defining property of the Rado complex, and ampleness is weaker than being able to extend any simplicial complex to any simplicial complex with one additional point.

4 Summary

Despite the additional structure of the Rado complex, it shares many more symmetries with the Rado graph than one might expect. In particular,

every regular group of automorphisms of the Rado graph corresponds to a regular group of automorphisms for the Rado complex. This is an interesting development in its own right, but this is by no means a complete classification of the automorphism group.

In my mind, the main impacts of my research are in showing that the automorphism group of the Rado complex is a rich and interesting object. If nothing else, it shows that studying this object is not a fruitless endeavour. As for potential next steps, one may be able to classify some of the non-regular automorphisms of the complex, but I think a particularly fruitful route may be to try to apply my method to learn more about the Rado graph itself. My method seems to suggest that a vast number of automorphisms correspond to each Cayley graph representation of the Rado graph, which could determine the size of some associated conjugacy classes.

5 Conclusion

As one could infer from the large amount of exposition at the beginning of this essay, one of the biggest challenges I faced during this project was acquiring the technical knowledge necessary to even understand the problem properly, since most of the definitions I have given above have not been covered in my degree. To give an example, characterising the n-skeletons requires an understanding of both the Rado graph and the use of Fraisse's theorem, which is a major theorem from model theory, which I am also unfamiliar with. Although I did begin my research by familiarising myself with these concepts, invariably I would realise that I had misunderstood something which would cause me to have to redo some aspects of my work.

Another challenge was that, due to the nature of the work, I could spend sometimes as long as a week without any progress, before a slight change in perspective would leave the problem seeming almost trivial. This was most apparent in the change of approach from creating a group of automorphisms that works for all n-skeletons, and instead creating a Cayley complex from a group of automorphisms. Although it is important to not give up on a strategy too soon, this mental flexibility is a very important skill, and one I think this project has helped to improve in me.

I will seek to publish a technical pre print on Arxiv soon, and if I have further success investigating the automorphism group then I will seek to submit this work to a journal.

I would like to thank Lord Laidlaw for providing me with the opportunity to carry out this work, as well as to my supervisor Peter Cameron, and to the whole Laidlaw team for their help over these stressful few months.

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