Enumeration, Symmetry and Classification of Fractal Carpets

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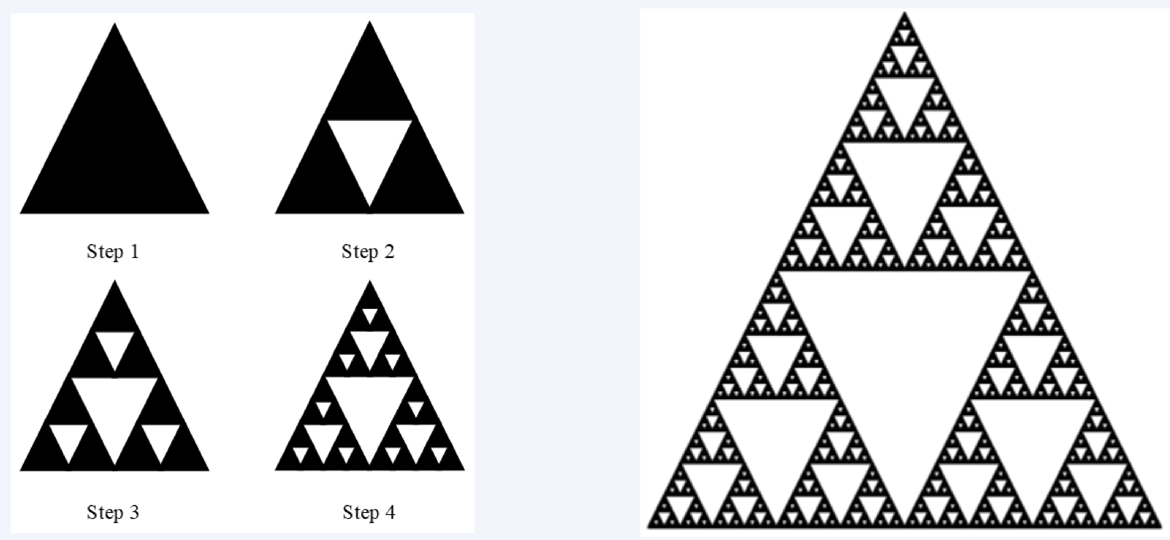
1. Introduction

Fractal geometry is a modern field in pure maths that was developed to study complicated mathematical structures with detail at arbitrarily fine scales that cannot be studied by classical means. Over the last few decades, the interest in this field has bloomed and is now one of the most active fields in mathematics. Fractals are not limited to mathematics, continuing to be applied in many other disciplines such as physical and biological sciences, economics, and art.

One way of producing a fractal is by an iterative procedure, as in Figure 1. At each stage an inverted equilateral triangle is removed from each of the solid triangles. This process continues ad infinitum to give the Sierpinski triangle. Note that the Sierpinski triangle is composed of many smaller scale copies of itself, a property called 'self-similarity'. Such fractals may be put into a more general context known as Iterated Function System (IFS) constructions [1].

This project focused on a particular family of self-similar fractals known as fractal carpets, their name coming from their square-based construction.

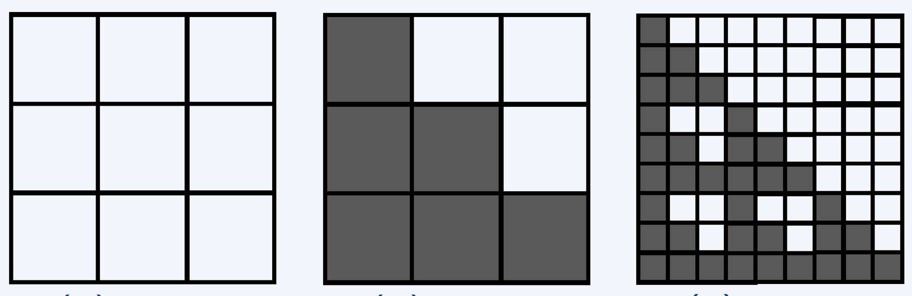


(a) Construction (b) Fractal

Figure 1: Sierpinski Triangle

The first part of the project applied a method to count all the possible different fractals that may be constructed from certain square-based IFS patterns. The second part investigated the connectivity of various IFS carpet constructions.

2. Enumeration and Symmetry



(a) Step 1 (b) Step 2 (c) Step 3

Figure 2: Right-angled Sierpinski Triangle Construction

We divide a square D into an $m \times m$ grid of smaller squares (Fig. 2a) and select some of these (Fig. 2b) to form a 'pattern'. At the next stage we replace each of the squares in the pattern by a scaled-down copy of the pattern itself (Fig. 2c). We now continue in this way, replacing squares by smaller and smaller scale copies of the pattern to get a fractal F (Fig. 3).

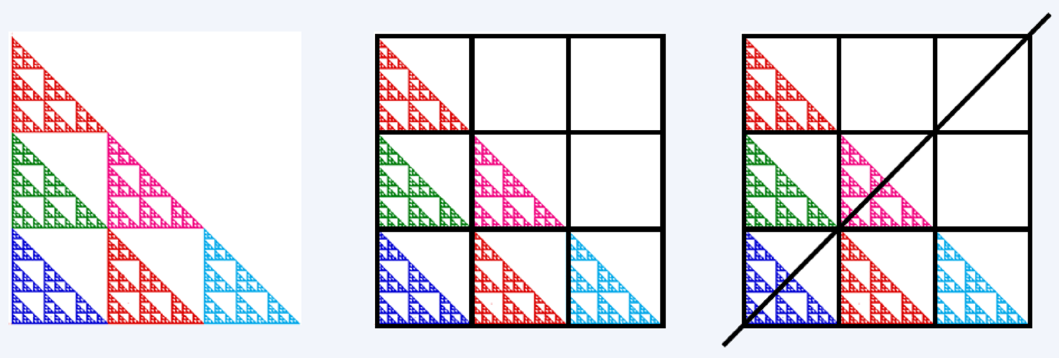


Figure 3: Right-angled Sierpinski Triangle

However, things are more complicated than this. When we replace each square with a small copy of the pattern we can choose to rotate or flip-over the copy as we do so. This gives rise to a large family of constructions. Our aim is to count the number of different fractals of each symmetry type.

Applying the method from [2], I found the number of attractors with each type of symmetry for fractals based on three new patterns, including the one in Figure 4.

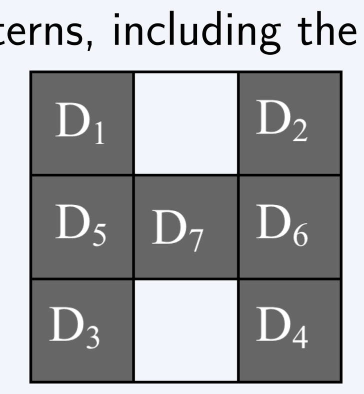
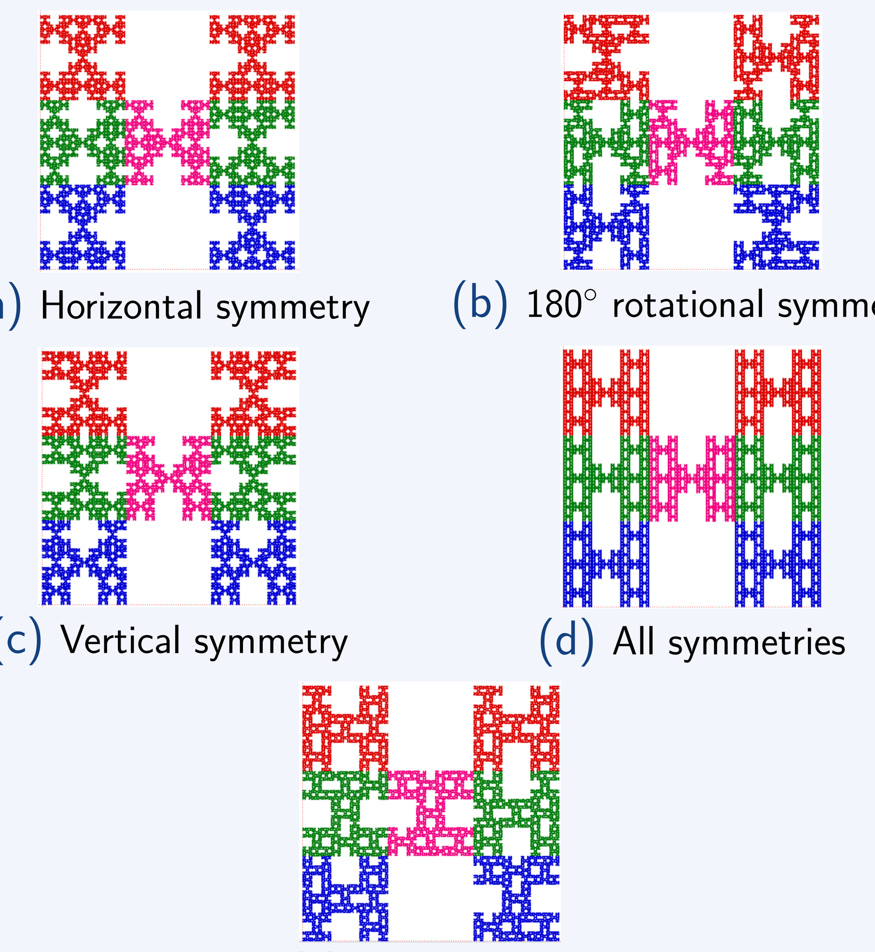


Figure 4: Example



(a) Horizontal symmetry (b) 180° rotational symmetry (c) Vertical symmetry (d) All symmetries (e) No symmetry

Figure 5: Attractors of varying symmetry

Here are pictures of 5 of the 1999108 different fractals based on the pattern in Figure 4. Note that each is made up of 7 one third scale copies of itself which may be reflected or rotated. The fractals display different forms of symmetry, for example that in Figure 5a has mirror symmetry about a horizontal line through the centre.

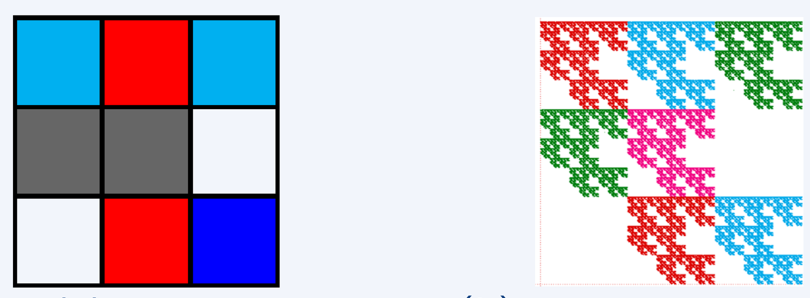
By a method that utilises group theory to analyse symmetries, I calculated the number of different fractals in each symmetry class that arise from the pattern in Figure 4. These are listed in Table 1.

Symmetry	Number of Fractals
All Symmetries	4
Horizontal	64
Vertical	64
180°	128
none	1998848

Table 1: Number of Fractals for each symmetry type

3. Connectivity and Classification

Fractal carpets can be classified in numerous ways such as by their topology or fractal dimension. This part of the project considered two aspects: when the carpets are connected and when they are 'dendrites', that is having a tree-like structure.

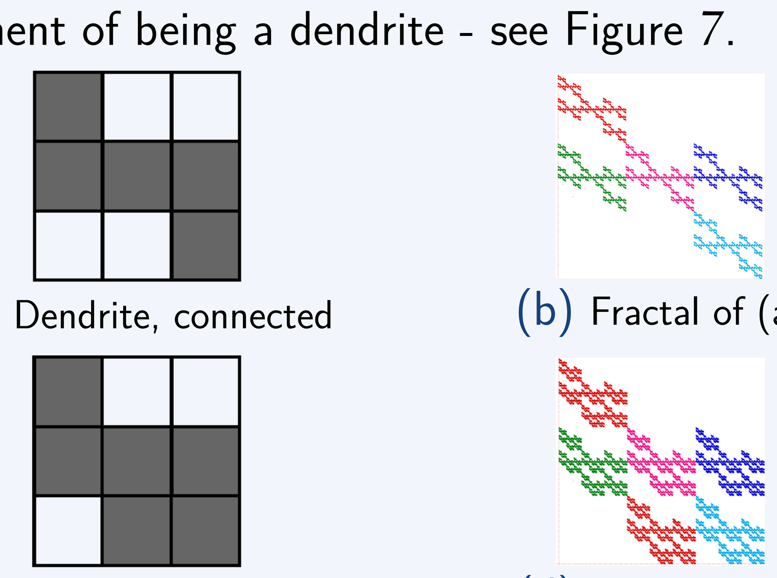


(a) Pattern (b) Corresponding Fractal

Figure 6: Example of a fractal carpet class

Figure 6 shows a pattern from one of the studied classes. By identifying key squares on the edge of the pattern we can ascertain how connectivity behaves when small copies are substituted in the pattern. In Figure 6, the light blue squares are the left and right exits, the red squares are the top and bottom exits, and the dark blue square is the horizontal mirror image of the right exit. The presence of such squares in a pattern ensures that when two such squares are placed together (with or without horizontal reflection) connectivity carries over to the subsquares.

A similar class was considered with the additional requirement of being a dendrite - see Figure 7.



(a) Dendrite, connected (b) Fractal of (a) (c) Not a Dendrite, connected (d) Fractal of (c)

Figure 7: Connectivity vs Dendrites

4. Conclusion

Using the method of Falconer and O'Connor [1] I have successfully found the number of distinct attractors for several new patterns. This work could be continued by writing an algorithm to automate the enumeration method. By generalising the arguments of Cristea and Steinsky [2] I have also shown that several new classes of fractal carpets are connected or are dendrites. Given the applicability of fractal geometry the methods used in this project may find application in other disciplines.

5. Acknowledgements

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References

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