Enumeration, Symmetry and Classification of Fractal Carpets

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1 Introduction

Fractal geometry is a modern field in pure maths that was developed to study complicated mathematical structures with detail at arbitrarily small scales that could not be studied by classical means. Over the last few decades, the interest in this field has bloomed and is now one of the most recognisable fields in the whole of mathematics. Its developments are not limited to mathematics, affecting other disciplines such as sciences, economics, and art.

To produce a fractal an iterative procedure is undertaken. An example of this is the Sierpinski triangle, as shown below:

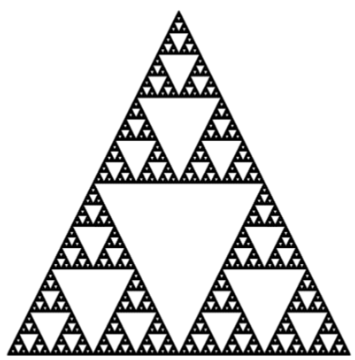


Figure 1: Sierpinski Triangle

In creating the Sierpinski triangle, we start with an equilateral triangle with unit side length. We then remove an equilateral triangle from the centre of this, which is horizontally flipped. We then do this to the three remaining triangles to get to the next step, and continue ad infinitum – this results with the Sierpinski Triangle. The second step of the construction could equivalently be viewed as mapping the unit triangle onto the smaller triangles by 3 separate contractions, namely $\{f_1, f_2, f_3\}$ - this is known as an Iterative function system (IFS). An attractor for an IFS is defined as a non-empty compact set, F, satisfying $F = \bigcup_i f_i(F)$, and by a common result in fractal geometry, such a set always exists and is unique for any IFS [1]. The attractors of IFSs are usually fractals such as in this example where the attractor is the Sierpinski triangle, a fractal that is made up of 3 copies of itself at scale $\frac{1}{2}$.

The research project focused on a particular family of self-similar fractals know as fractal carpets, with their name coming from their square like construction. What was studied can be split into two sections. The first section looks into methods in group theory used to study the number of the attractors of various iterated function systems with particular symmetries. The second section looks into the connectivity of various constructions and investigates several classes of fractal carpets.

2 Enumeration and Symmetry

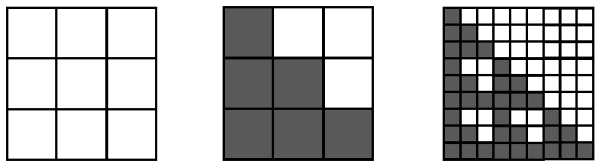


Figure 2: Right-angled Sierpinski Triangle Construction

The first stage in the construction of a fractal square is done by taking the unit square, D, dividing it into an $m \times m$ grid and shading a selection of subsquares, namely $\{D_1, D_2, \ldots, D_n\}$. For each subsquare there is a contraction, $f_i : D \to D_i$, which maps the unit square onto the subsquare, and this family of contractions gives the IFS of the fractal carpet. Each contraction in the IFS, f_i, could either map D directly onto D_i, or map D with a reflection/rotation onto D_i. As both D and D_i are squares, the reflections/rotations that f_i can induce are in the dihedral group of order 8. That means there are 8^n possible IFSs for any given selection of n subsquares in the $m \times m$ grid. For a particular selection of subsquares, say $\{D_1, D_2, \ldots, D_n\}$, some of the IFSs will have the same attractor. We define $\mathcal{C}(\mathrm{D})$ to be the set containing all IFS $I_j = \{f_1, \ldots, f_n\}$ such that $f_i : D \to D_i$ where f_i is a similarity. The following method is taken from a research paper [2] which allows us to enumerate the attractors of particular types of symmetry (e.g. horizontal, 180 rotational, etc). Attractors without symmetry will have a unique IFS.

The attractor for figure 2 is shown in figure 3 and has diagonal symmetry, i.e. reflects into itself by a 'mirror' on the leading diagonal.

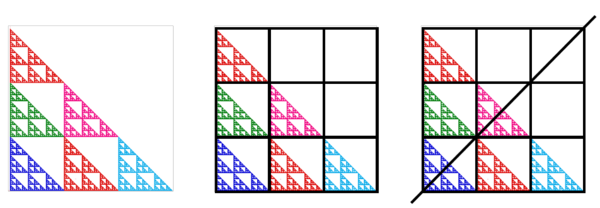


Figure 3: Right-angled Sierpinski Triangle

For the following examples, refer to the Notes section at the end of the essay for the relevant terminology and results.

Using these results (in particular Result 4) a procedure can be undertaken to enumerate all the attractors of various symmetries. A slight adaptation from Result 4 is used when an attractor satisfies $H \leq H'$ and some other $K \leq K'$. I did this for four examples, three of which had not been studied before, and two of them are given below:

Example 1: The method for this example can be used for other examples with slight adjust-

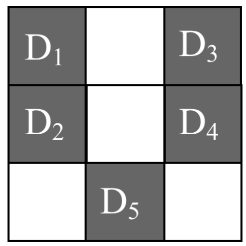


Figure 4: Example 1

ments.

Let D be the unit square and $D_1, \dots, D_5$ be the subsquares. Let $f_i : D \to D_i$ be a similarity mapping of ratio $\frac{1}{3}$. Let E_j be the attractor of the IFS $I_j = \{f_1, \dots, f_5\}$ so that $E = \bigcup_i f_i(E)$. Then they're 8 possible choices for each f_i, giving $8^5 = 32768$ different IFSs.

The attractors either have vertical symmetry (i.e. reflective symmetry about a vertical line through the centre of the square) or no symmetry. Let G_0 be the set of symmetries of D and G be the set of symmetries of $\bigcup_i D_i$. Then $G_0 =$ dihedral group of order 8, and $G = \{i, \sigma\}$ where i is the identity and σ is a vertical reflection along the centre of the unit square, D.

We look for IFSs that satisfy $H \leq H'(\{f_i\})$, where $H = G = \{i, \sigma\}$ and $\{f_i\} = \{f_1, \dots, f_5\}$. This can be done using the results from [2] to construct table 1. Analysing this table con-

Orbit of H	$\text{stab}_H j$	$f_j^{-1}(\text{stab}_H j) f_j \leq H$	$f_i \in h f_j H$
{1,3}	$\{i\}$	$f_1 \in G_0$	$f_3 \in h f_1 H$
{2,4}	$\{i\}$	$f_2 \in G_0$	$f_4 \in h f_2 H$
{5}	$\{i, \sigma\}$	$f_5^{-1}\{i, \sigma\} f_5 \leq H$	

Table 1: Results for example 1

cludes they are $8 \cdot 8 \cdot 2 \cdot 2 \cdot 4 = 1024$ possible IFSs that satisfy $H \leq H'(\{f_i\})$. Each attractor with vertical symmetry will be the attractor to $2^5 = 32$ IFSs. That means they're $\frac{1024}{2^5} = 32$ different attractors with vertical symmetry. The remaining attractors will have no symmetry and will each correspond to one IFS: they are $32768 - 1024 = 31744$ such IFSs.

Example 2: For this example, the symmetry group of $\bigcup_i D_i$ is the dihedral group of or-

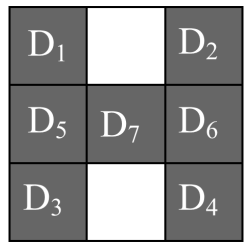


Figure 5: Example 2

der 4, denoted by Q, where i is the identity, σ is a horizontal reflection and ρ is a $\frac{\pi}{2}$ rotation. We denote M as the dihedral group of order 8. First we individually consider all the IFSs satisfying $Q' \geq Q$, $H' \geq H$, $V' \geq V$, and $P' \geq P$ where $H = \{i, \sigma\}$, $V = \{i, \rho^2\sigma\}$, and $P = \{i, \rho^2\}$ - note the composition of functions is read from right to left. We do this by using the method shown in Example 1. A slight change is needed however.

Looking at vertical symmetry, V, the procedure used in Example 1 gives all the IFSs, I_j, satisfying $V' \geq V$. These will have sym$E_j \geq V$ (from result 3), where E_j is the attractor of I_j. Some attractors will satisfy sym$E_j = V$ and others will satisfy sym$E_j = Q$. It is worth noting that for the procedure that gives all IFSs satisfying $Q' \geq Q$, not all of these IFSs will satisfy $V' \geq V$ as well, even though they all have attractors satisfying sym$E_j \geq V$. If we consider attractors only satisfying $V = \text{sym}E_j$, we need to look at all the IFSs satisfying $V' \geq V$ and remove ones with sym$E_j > V$. As no subgroup of Q has order 3, we need only consider removing IFSs that satisfy $V' \geq V$ and $Q' \geq Q$. This is done by comparing each similarity, f_i, in the two tables for $V' \geq V$ and $Q' \geq Q$ produced using the method in example 1, and creating a combined table of the two - this is shown in table 2 (where $B(1)$ is $f_j^{-1}(\text{stab}_Q j)f_j \leq Q$, C(1) is $f_i \in qf_jQ$, B(2) is $f_j^{-1}(\text{stab}_V j)f_j \leq V$, and C(2) is $f_i \in vf_jV$). Horizontal and rotational symmetry can be checked using the same method.

Orbit of Q	$stab_Q j$	$B(1)$	Orbit of V	$stab_V j$	$C(1)$	$B(2)$	C(2)
{1,2,3,4}	$\{i\}$	$f_1 \in M$	{1,2}	$\{i\}$		$f_1 \in M$	$f_2 \in vf_1V$
			{3,4}	$\{i\}$	$f_3 \in mf_1M$		$f_4 \in vf_3V$
{5,6}	H	$f_5 \in M$	{5,6}	$\{i\}$		$f_5 \in M$	$f_6 \in vf_5V$
{7}	Q	$f_7 \in M$	{7}	V		$f_7 \in Q$	

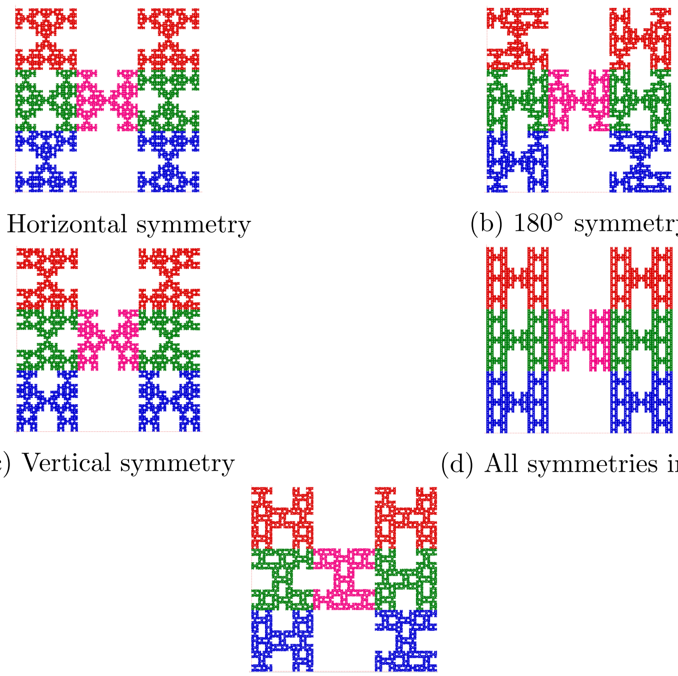
Table 2: IFSs satisfying $V' \geq V$ and $Q' \geq Q$

Using the method above and in Example 1, it was found that there was 1999108 distinct attractors for Example 2. The number of attractors for each type of symmetry is noted on table 3. A program was then used to construct attractors of each type of symmetry and is shown in figure 6.

These ideas can be extended to many other constructions, such as hexagon or 3D cube

Symmetry	Q	Horizontal	Vertical	180°	none
Order	4	2	2	2	1
Number of attractors	4	64	64	128	1998848

Table 3: Number of attractors for each symmetry type



(a) Horizontal symmetry

(b) 180° symmetry

(c) Vertical symmetry

(d) All symmetries in Q

(e) No symmetry

Figure 6: Attractors of varying symmetry types for Example 2

constructions [2].

3 Connectivity and Classification

An intriguing and complex question is to establish whether IFS attractors are connected. Here we present a new class of IFS carpets where the IFS maps may involve reflections, and must have connected attractors. We define a m-pattern to be a collection of squares from the $m \times m$ subdivision of the unit square: refer to the Notes section at the end of the essay for the rest of the relevant terminology.

Definition 1. *For $m \geq 2$, we say an m-pattern P is a class $\mathcal{C}(m)$ if*
1. P is connected by squares that share a side

2. At least one square in the top row is directly above one square in the botton row – so at least one top and bottom exit.
3. At least on square in the far left column is in the same row as one square in the far column row – so at least one left and right exit.
4. P contains the horizontal mirror image of the left or right exit (e.g. horizontal image of left or right squares satisfying 3)

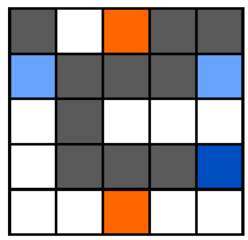


Figure 7: Example satisfying definition 1, for $\mathcal{C}(5)$

Figure 7 shows a pattern that satisfies definition 1 for $\mathcal{C}(5)$. The light blue squares are the left and right exits, the orange squares are the top and bottom exits, and the dark blue square is the horizontal mirror image of the right exit. Figure 8 shows two 5-patterns connectivity. We consider a 5-pattern, P, to be edge-connected if the squares in P all share a side with another square in P.

(a) Connected

(b) Disconnected

Figure 8: Connectivity of two 5-patterns

Lemma 1. Let E_n be an $m^n \times m^n$-pattern obtained by repeated substitution of E_1 in itself, as defined in $\star$ from the notes section, and let P be an m-pattern such that $P \in \mathcal{C}(m)$. Then $E_n \in \mathcal{C}(m^n)$, for all $n \in \mathbb{N} \setminus \{0\}$.

Proof. We use induction on n. The base case holds true with $P = E_1$. We assume $E_n \in \mathcal{C}(m^n)$ for our inductive hypothesis. Consider E_{n+1}.

For Property 1 of Definition 1: Let $S, S' \in E_{n+1}$ be two distinct subsquares, and let $T, T' \in E_n$ be the subsquares in E_n that S and S' are contained in, respectively. As E_n is our inductive hypothesis, is it connected and so there exists a sequence of squares in E_n from T to T'; $T = T_0, T_1, \ldots, T_w = T'$, where $T_i \in E_n$, for all $i \in \{0, 1, \ldots, w\}$. This sequence is such that T_{i-1} and T_i have a common side, where $1 \leq i \leq w$. For $0 \leq i \leq w$, $T_i \bigcap E_{n+1}$ is a scaled down

copy of P, either directly or with a horizontal reflection, and thus $T_i \bigcap E_{n+1}$ satisfies all the properties. If T_{i-1} and T_i for $1 \leq i \leq w$ are horizontally adjacent to each other, they have subsquares in E_{n+1} that share a common side, namely $S^-_{i-1} \in T_{i-1}$ and $S^+_i \in T_i$, by properties 3 and 4. The same result follows if T_{i-1} and T_i are vertically adjacent, by Property 2.
For $1 \leq i \leq j-1$, S^-_i is connected to S^+_i by a chain of edge-adjacent subsquares, sharing common edges, in $T_i \bigcap E_{n+1}$. S is connected to S^-_0 by a chain of edge-adjacent subsquares in $T \bigcap E_{n+1}$, and S^+_j is connected to S' by a chain of edge-adjacent subsquares in E_{n+1}, between S and S', namely $S, \ldots, S^-_0, S^+_1, \ldots, S^-_1, S^+_2, \ldots, S^-_2, \ldots, S^+_j, \ldots, S'$, thus the m^{n+1}-pattern, E_{n+1}, satisfies property 1.

For Property 2 of Definition 1: Given $E_n \in \mathcal{C}(m^n)$, it satisfies Property 2, so there exists $T, T' \in E_n$ such that T is in the top row of E_n and T' is directly below T and is in the bottom row of E_n. Let $\phi_1 : [0,1]^2 \to T$ and $\phi'_1 : [0,1]^2 \to T'$ be direct similarities and let $\phi_2 : [0,1]^2 \to T$ and $\phi'_2 : [0,1]^2 \to T'$ be similarities with a horizontal reflection. Given $P \in \mathcal{C}(m)$, there exists $S, S' \in P$ such that S is in the top row of P and S' is directly below S in the bottom row of P. Then $\phi_1(S)$ or $\phi_2(S')$, and $\phi'_1(S')$ or $\phi'_2(S)$ are squares in E_{n+1} in the top and bottom rows respectively, with $\phi_1(S)$ or $\phi_2(S')$ in the top row and $\phi'_1(S')$ or $\phi'_2(S)$ in the bottom row, directly beneath it. Thus E_{n+1} satisfies property 2.

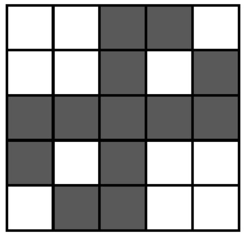
For Property 3 and 4 of Definition 1: Given $E_n \in \mathcal{C}(m^n)$, there are squares $T, T', T'' \in E_n$ with T in the same row as T', T is adjacent to the left side of $[0,1]^2$ and T' is adjacent to the right side of $[0,1]^2$. Without loss of generality say T'' is the horizontal mirror image of T'. Let $\varphi_1 : [0,1]^2 \to T$, $\varphi'_1 : [0,1]^2 \to T'$, and $\varphi''_1 : [0,1]^2 \to T''$ be the direct similarity maps, and let $\varphi_2 : [0,1]^2 \to T$, $\varphi'_2 : [0,1]^2 \to T'$, and $\varphi''_2 : [0,1]^2 \to T''$ be the similarity maps that also induce a horizontal reflection. As $P \in \mathcal{C}(m)$, there are squares S, S', and $S'' \in$ P with S in the far left column, and S' in the far right column, such that S and S' are in the same row. The square S'' is in the same column as S' and is it's horizontal mirror image. Then $\varphi_1(S)$ or $\varphi_2(S)$ is in E_{n+1}, and $\varphi'_i(S'), \varphi'_i(S'')$, and $\varphi''_j(S'), \varphi''_j(S'')$ are also in E_{n+1} for i, j $\in \{1,2\}$. Without loss of generality we can say $\varphi_2(S)$ is in E_{n+1} and $i=1, j=2$. Then $\varphi_2(S)$ is adjacent to the left side of $[0,1]^2$ and in the same row as $\varphi'_1(S'')$ which is adjacent to the right side of $[0,1]^2$. Also $\varphi''_2(S'')$ is in the same column as $\varphi'_1(S'')$ and its horizontal mirror image. Thus E_{n+1} satisfies Properties 3 and 4. □

The following lemma is standard in topology and so is just stated [4]:

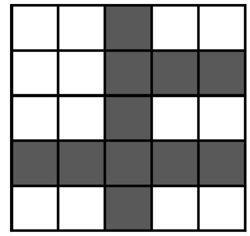
Lemma 2. Let $F_1 \supseteq F_2 \supseteq F_3 \supseteq \ldots$ be a decreasing sequence of non-empty closed bounded subsets of $\mathbb{R}^N$ where $N \in \mathbb{N} \setminus \{0\}$. Then $F = \bigcap_{n=0}^{\infty}$ is non-empty and closed. If F_i is connected for all i, then F is connected.

Theorem 3. Let P be a m-pattern and I be an IFS as defined at the basic set up section. If $P \in \mathcal{C}(m)$, then the attractor F of I is connected.

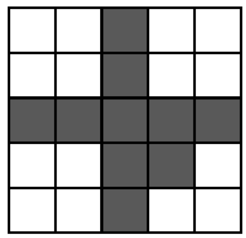
Proof. From lemma 1, E_n as defined in $\star$ is connected for all n. It is clear that $E_n \supset E_{n+1} \supset \ldots$ is a decreasing sequence, and so $F = \bigcap_{n=0}^{\infty} E_n$ is non-empty and compact. By Lemma 2, it follows that F, the attractor of I, is connected. □



(a) Dendrite, connected



(b) Dendrite, connected



(c) Not a Dendrite but is connected

Figure 9: Connectivity vs Dendrites

For similar classes, we can get similar results with a slight adjustment to the proof (e.g. if the vertical image was included instead of horizontal one). The class for Definition 1 was looked at again, this time with the condition that it needed to be a dendrite — see Figure 9 — which means you can only get from one square to another in one way. This was done with 7 properties in total to ensure a similar result of Lemma 1 for this class would hold: the proof was slightly different given the attractor must be a dendrite, and was a generalisation of the proof given in the first lemma of [3].

4 Conclusion

Through the analysis of the paper of Falconer and O'Connor [2], and studying the proofs of Cristea and Steinsky [3] I have successfully found the number of attractors for four IFSs and proved several results for two classifications of fractal carpets. This has given me a deeper understanding of research in pure mathematics. I have improved my time-management and ability to multi-task, as well as acquired discipline and resilience in my problem-solving skills. This project could be extended by making an algorithm in GAP of the method used in the 'enumeration and symmetry' section, and given the applicability of fractal geometry, the methods used in this project may have application in other disciplines.

5 Acknowledgements

I thank the Laidlaw Foundation and Lord Laidlaw himself for giving me the opportunity to be part of this excellent program. I also thank my supervisor, Professor Kenneth Falconer, for his patience and dedication in helping me through this research project.

6 Notes

Terminology and Results needed for section 2:
Let $\text{sym}(E_j)$ be the symmetry group of the attractor for $I_j \in \mathcal{C}(\text{D})$, and let G_0 be the symmetry group for D and G be the symmetry group for $\bigcup_i D_i$, for $D_i \in \{D_1, \ldots, D_n\}$. We denote g^* as the permutation that $g \in G$ induces on the set $\{D1, \ldots, D_n\}$ and it is of the form (1 2 3 ... n), e.g. (2 3 1 4). Let H be a subgroup of G, and denote an IFS $\{f_1, \ldots, f_n\} = \{f_i\}$. Then define $H' = H'(\{f_i\}) = \{g \in G : f^{-1}_{g^*(i)} g f_i \in H \ \forall i\}$.
The orbit of j under H is defined as $O = \{i : i = h^* j$ for some $h \in H\}$, and the stabilizer of $j \in H$ is defined as $\text{stab}_H j = \{h \in H : h^* j = j\}$

Results[2]:
Result 1: $\{f_1, \ldots, f_n\}$ and $\{f'_1, \ldots, f'_n\} \in \mathcal{C}(\text{D})$ have the same attractor E if and only if $f'_i \in f_i \ \text{sym}E$

Result 2: Let E be the attractor of the IFS $\{f_1, \ldots, f_n\} \in \mathcal{C}(D)$. Then $g \in \text{sym}E$ if and only if $f^{-1}_{g^*(i)} g f_i \in \text{sym}E$, for all i.

Result 3: Let $\{f_1, \ldots, f_n\} = \{f_i\}$ be an IFS with attractor E. Then,
(i) There exists a k such that, $G > G' > G'' > \ldots > G^k = G^{k+1} = \ldots = \text{sym}E$. Then $\text{sym}E = (\text{sym}E)'$ so $\text{sym}E \leq (\text{sym}E)'$
(ii) $\text{sym}E = H$ where H is the maximal subgroup of G such that $H \leq H'(\{f_i\})$.

Result 4: $H \leq H'(\{f_i\})$ if and only if for every orbit O under H, for some chosen $j \in O$, $f_j^{-1}(\text{stab}_H j) f_j \leq H$, and for all other $i \in O$, $f_i \in h f_j H$ for any $h \in H$ such that $i = h^* j$.

Set up needed for section 3: Consider a set P to be a union of subsquares of the unit square of the form $[\frac{a-1}{m}, \frac{a}{m}] \times [\frac{b-1}{m}, \frac{b}{m}]$ such that $1 \leq a, b \leq m$ and $a, b \in \mathbb{N}$, and let the cardinality of P be greater than 1. We define such a set, P, to be an m-pattern as each subsquare has side length $\frac{1}{m}$ for $m \in \mathbb{N}$.

Define $I = \{f_1, \ldots, f_k\}$ to be an IFS such that $f_i : [0,1]^2 \to P$ for all i, where f_i is either a direct similarity mapping the unit square onto a subsquare of P, or f_i is a similarity mapping that induces a horizontal reflection of $[0,1]^2$ and then maps it onto P.

Define $E_n = \bigcup_{1 \leq i_j \leq k} f_{i_1} \circ \ldots \circ f_{i_n}([0,1]^2) \star$
Then E_n is obtained by mapping P onto itself, either directly or with a horizontal rotation, $n-1$ times (Given E_1 is obtained by mapping $[0,1]^2$ onto P, thus $E_1 = P$). Thus E_n is a m^n-pattern.

By general theory in fractal geometry, $E = \bigcap_n E_n$ is the unique attractor of I, and is non-empty and compact.

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