Free Commutative Monoids Arising as the Bimorphism Monoids of Infinite Graphs

Isaac Dilley

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1 Introduction

1.1 Definitions

When we refer to a graph, G, we mean an ordered pair (V, E), where V is a set of vertices, and $E \subseteq [V]^2$ is a set of edges. This project focused on properties specific to infinite graphs, but a great general resource on graphs can be found in [1].

A bimorphism of a graph G is a bijection $\alpha : V(G) \longrightarrow V(G)$ such that $v \sim w$ implies $v\alpha \sim w\alpha$. We see that bimorphisms preserve edge relations. It is not necessarily true that bimorphisms preserve non-edges. A bimorphism that preserves non-edges is called an automorphism [2].

The bimorphisms of an infinite graph form a monoid. For an overview of semigroups and monoids, see [3], but for this project we make use of the following definitions. A monoid is a set M equipped with a binary operation, $\bullet : M \times M \longrightarrow M$. This operation must be associative, meaning for all a, b, c in M, the statement $(a \bullet b) \bullet c \ = \ a \bullet (b \bullet c)$ must hold. There must also exist an identity element e in M such that $a \bullet e = e \bullet a = a$ for all a in M. If M has elements a and b such that $a \bullet b = b \bullet a = e$, then a and b are inverses. The set of all invertible elements in M is called the group of units of M. When considering the bimorphism monoid of a graph, its group of units is its automorphism group.

A monoid M can be embedded in a group $\mathcal{G}$ if there exists an injective homomorphism $f : M \longrightarrow \mathcal{G}$. We note that as each bimorphism is a permutation of vertices, the bimorphism monoid of a graph with vertex set V is a submonoid of the permutation group on its vertex set, $\mathrm{Sym}(V)$. Therefore, every bimorphism monoid is a group-embeddable monoid. Another group-embeddable monoid we consider is the free commutative monoid on a set. Given a set X, a free commutative semigroup F_X can be formed by taking all non-empty sums of positive integer multiples of elements of X. From F_X, we can form the free commutative monoid by appending an identity element. The non-negative integers under addition is an example of a free commutative monoid, specifically the free commutative monoid on a single generator. The set of positive integers under multiplication is also a free commutative monoid, with the generating set being the set of all prime numbers. For more on free commutative monoids, see [4]

1.2 Characterizing group-embeddable monoids

The question of when a monoid embeds in a group is one of historical significance. In 1939, Mal'cev proved that a system of necessary and sufficient conditions for a monoid to embed in a group cannot be finite, and gave the first system of conditions. Lambek later proved his polyhedral system of conditions for a semigroup to embed in a group [5, 6]. This system of conditions allowed for a geometric interpretation. To my knowledge, these are the only proved systems of necessary and sufficient conditions for a monoid to be embeddable in a group.

When my supervisor began investigating bimorphism monoids of graphs during his doctoral thesis, he noted that every bimorphism monoid was group-embeddable, but he also wondered if every group-embeddable monoid arose as the bimorphism monoid of a graph. If this were the case, not only would it provide a new system of conditions for a monoid to be group-embeddable, it would also be a generalization to a famous theorem by Frucht, which states that every finite group arises as the automorphism group of a finite graph.

1.3 A graph with a familiar bimorphism monoid

In the world of group-embeddable monoids, it would be a challenge to find one more agreeable than $(\mathbb{N}_0, +)$, the non-negative integers under addition. Because no positive integers have inverses in $\mathbb{N}_0$, it is certainly not a group. However, the idea of embedding $\mathbb{N}_0$ in the group $\mathbb{Z}$ seems reasonable, and as it turns out, the monoid structure of $\mathbb{N}_0$ is entirely preserved, even in the presence of negative numbers.

My supervisor sought to construct a graph whose bimorphism monoid was isomorphic to $(\mathbb{N}_0, +)$. This led him to the 'incomplete ladder graph' found in Figure 1.

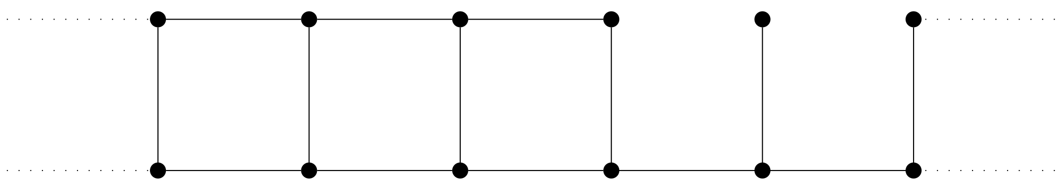

Figure 1: Incomplete ladder graph

If we restrict permutations of vertices to those that preserve both edges and non-edges, we find that the last complete 'rung' of the ladder has a vertex of degree two. Every automorphism must map this vertex to itself, since there are no other vertices of degree two. One of the neighbors of the degree two vertex has a path of length two to a degree one vertex, so that must also be mapped to itself. From there, it follows than an automorphism must map every vertex to itself, meaning the graph is rigid, and has a trivial automorphism group. If we restrict ourselves to permutations that only preserve edges, we would find that the only non-identity bimorphisms are those that shift every vertex evenly to the left. My supervisor proved this, and showed that because this monoid could be generated by the permutation that maps every vertex to the vertex immediately to its left, it was the free (commutative) monoid on a single generator. Therefore, the bimorphism monoid of this graph is isomorphic to $(\mathbb{N}_0, +)$.

2 Extension to the free commutative monoid on two generators

In my summer of research, I aimed to extend the construction my supervisor used for the free commutative monoid on one generator to free commutative monoids on an arbitrary number of generators. When my supervisor introduced me to the graph in Figure 1, he pointed out that the sides of the ladder contained the Cayley graphs of both $\mathbb{Z}$ and $\mathbb{N}_0$. A Cayley graph is another way of expressing the structure of a group or monoid, where the vertices are elements of the group or monoid, and edges (although typically colored and directed) show the action of generators on elements in the group or monoid. Cayley graphs are used extensively across group and semigroup theory to aid in visualizing structure [7]. The Cayley graph of $\mathbb{N}_0$ can be seen as the one ended number line on the top of the graph, and the Cayley graph of $\mathbb{Z}$ as the infinite number line on the bottom. My supervisor wondered if this idea of 'gluing' the Cayley graph of the monoid onto the Cayley graph of the group in which it embeds could be applied to other monoids.

I decided to approach the two generator case by using an infinite grid on the bottom, and a one-cornered grid on top. This seemed like a reasonable extension, as it used the Cayley graphs of $\mathbb{Z}^2$ and $\mathbb{N}_0^2$. However, an infinite grid has many more symmetries than an infinite line. If I wanted the graph to be rigid, I needed to glue the upper and lower grids together in a way that prevented non-identity automorphisms. I found that it was necessary for each point in the upper grid to be adjacent to two points in the lower grid. In the end, I was left with the graph seen in Figure 2.

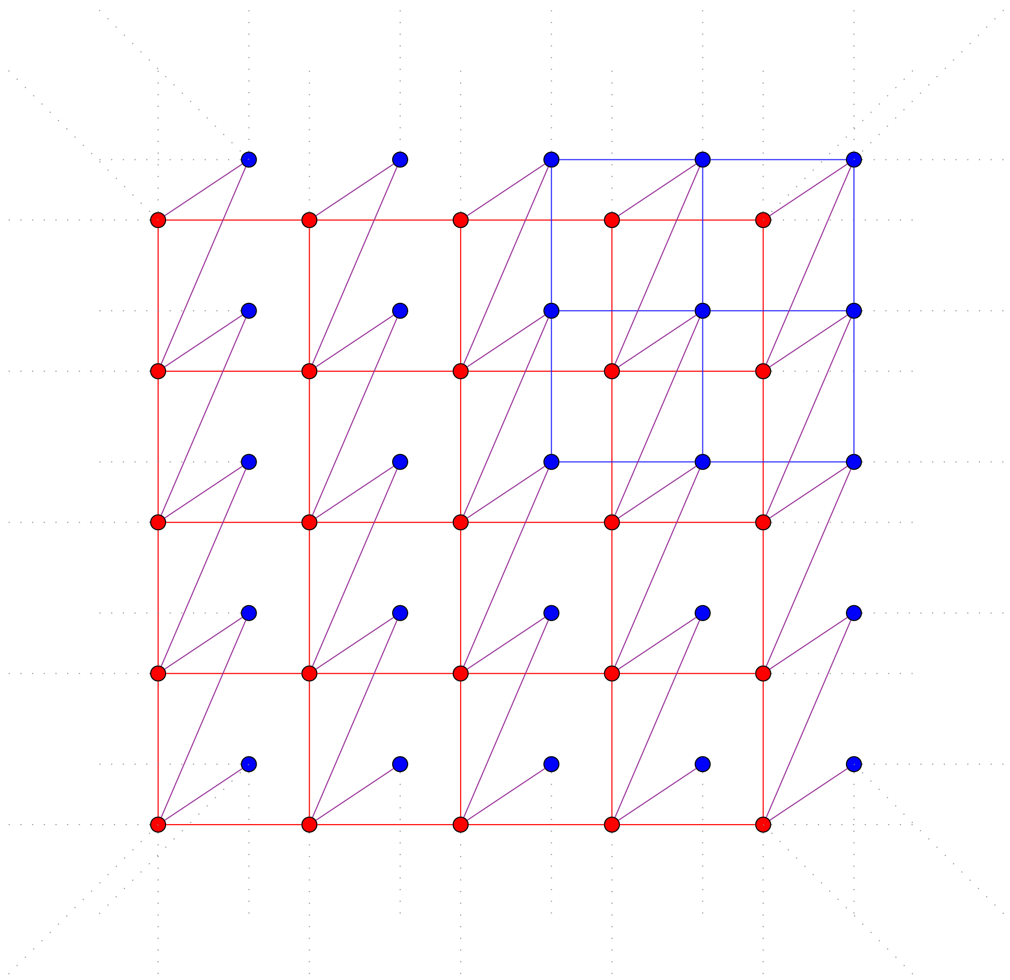

Figure 2: Table graph

In order to show that the bimorphism monoid of my table graph was isomorphic to $\mathbb{N}_0^2$, I needed to show that the only valid bimorphisms were translations that moved vertices up or to the right, and the identity automorphism. Then, I needed to prove that the bimorphism monoid described above is actually the same as the monoid of ordered pairs of non-negative integers under addition.

2.1 Sketch of proof

The proof follows from many little proofs, with each little proof eliminating possible candidates for bimorphisms. In the first little proof we show that valid bimorphisms cannot map the red vertices from the lower grid to the blue vertices in the upper grid. We use the fact that the corner is degree 4 to show that only blue vertices can map to it. From that, we know that the red adjacencies of that vertex must map to the red adjacencies of the corner, and the rest of the proof falls into place like a row of dominoes. The reason we prove this first is so we can use the fact that the restriction of a bimorphism to the lower grid will be a bimorphism of the lower grid. Because every vertex in the lower grid is degree 6, non-edges are trivially preserved by a bimorphism of the lower grid, and we have the more useful fact that the restriction of any bimorphism to the lower grid will be an automorphism of the lower grid. We also see that every vertex in the upper grid is connected to two vertices in the lower grid, so an automorphism of the bottom grid can only extend to one possible bimorphism of the graph.

We can now focus our attention on the automorphisms of a two dimensional grid graph. These include reflections (across a vertical axis, horizontal axis, or diagonal axis) rotations about a fixed point (not necessarily a vertex) in increments of $\frac{\pi}{2}$ radians, and translations. In the case of the reflections, we would pick an arbitrary axis and show that reflecting across that axis led to a blue or purple edge breaking. For non-identity rotations, the strategy is more of the same.

We now turn to translations. We begin by proving that a translation that maps vertices down or to the left cannot be a bimorphism, since the corner vertex will have at least one adjacency in the upper grid mapped to a non-adjacency.

We now aim to prove that the translation mapping each vertex to the vertex immediately above it, and the translation mapping each vertex to the vertex immediately right of it, are bimorphisms. We need to prove that every edge is preserved by these translations. We can again use the fact that the restriction of any translation to the lower grid will be an automorphism, and will necessarily preserve red edges. Because translations move the whole graph without altering orientation, the purple edges will also be preserved. When we consider how these translations affect adjacencies in the upper grid, we see that all edges are preserved, but each translation adds edges, so these bimorphisms are not automorphisms. We then use the fact that the bimorphisms of a graph form a monoid, meaning the composition of two bimorphisms is also a bimorphism, to prove that bimorphisms generated by these two translations is a subsemigroup of the bimorphism monoid of the table graph.

A quick check shows the semigroup generated by these two translations is commutative, and each bimorphism in this semigroup can be expressed uniquely up to order of terms as a product of positive powers of the two translations. This invites a natural isomorphism from this semigroup to the free commutative semigroup on two elements. Since we have exhausted all possible automorphisms of the lower grid, we have in fact exhausted all possible bimorphisms of the table graph, and we can confidently say that the only other bimorphism of our graph is the identity automorphism. We conclude that the bimorphism monoid of the table graph is in fact the free commutative monoid on two elements.

3 Extension to the free commutative monoid on n generators

While my initial goal with this project was to find a construction for the more general free commutative monoid on an arbitrary number of generators, the approach used to prove the two generator case becomes impractical very quickly as the number of generators increases. If we continued the method of gluing cayley graphs together, the three generator case would require gluing an infinite three dimensional grid to a single cornered cube. At only three generators, one of the most helpful tools in graph theory, visualization, becomes significantly less useful. Even if visualization wasn't an issue, an infinite three dimensional grid has far more symmetries to rule out, and any truly general case would require a more general approach than ruling out each case individually.

That being said, being able to extend the one generator case to two certainly offers insight into what a general construction could look like.

3.1 Possible construction for the n generator case

Let Γ be a graph with vertex set $V(\Gamma) = \mathbb{Z}^n \times \{0, 1\}$, and edge relations given by:

- $(v_1, v_2, \ldots, v_n, 0) \sim (w_1, w_2, \ldots, w_n, 0)$ if and only if
$$\sum_{i=1}^{n} |\, v_i - w_i \,| = 1$$
- $(v_1, v_2, \ldots, v_n, 1) \sim (w_1, w_2, \ldots, w_n, 1)$ if and only if $v_1, w_1, ..., v_n, w_n \geq 0$ and
$$\sum_{i=1}^{n} |\, v_i - w_i \,| = 1$$

- $(v_1, v_2, \ldots, v_n, 0) \sim (w_1, w_2, \ldots, w_n, 1)$ if and only if $v_n = w_n$ and if there exists an i such that $v_i \neq w_i$, then we require that $w_i - v_i = i$, and $v_j = w_j$ for all $j \neq i$.

This graph construction gives a lower n-dimensional grid and an upper single cornered n-cube, with the corner at $(0, \ldots, 0, 1)$. Each vertex in the upper grid is glued to n points in the lower grid in an asymmetric manner. The necessity of this can be seen by the diagonal purple edges in the table graph of Figure 2. Without the diagonal edges, a reflection across the plane bisecting the corner would result in an automorphism. As we increase the dimension of the underlying grid, we must also increase the number of connections between grids to restrict possible symmetries.

While the above construction is by no means proved to have a bimorphism monoid isomorphic to the free commutative monoid on n generators, I believe that the method of gluing cayley graphs together in some manner is particularly well suited for commutative group-embeddable monoids. Hopefully, more exploration will not only lead to a proof of the above construction, but will further our understanding of the class of group-embeddable monoids as a whole.

4 Acknowledgments

Going into this project, I had no idea what to expect from research in mathematics. I didn't want to set my expectations too high, but I also wanted an opportunity to challenge myself and tackle a difficult problem. However, none of that would have been possible in the first place without the generous support of Lord Laidlaw and the Laidlaw Foundation.

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Finally, to my supervisor, Dr Tom Coleman, thank you for going above and beyond to ensure this research experience was as rewarding as possible for me. I hope we have the opportunity to continue exploring the world of group-embeddable monoids this year.

References

[1] Reinhard Diestel. Graduate texts in mathematics. *Graph theory*, 173, 2000. 1

[2] Thomas DH Coleman, David M Evans, and Robert D Gray. Permutation monoids and mb-homogeneity for graphs and relational structures. *European Journal of Combinatorics*, 78:163–189, 2019. 1

[3] John Mackintosh Howie. *Fundamentals of semigroup theory*. Number 12 in London Mathematical Society monographs. Oxford University Press, 1995. 1

[4] Pierre Antoine Grillet. Commutative semigroups, volume 2 of. *Advances in Mathematics (Dordrecht)*, 2001. 2

[5] Joachim Lambek. The immersibility of a semigroup into a group. *Canadian Journal of Mathematics*, 3:34–43, 1951. 2

[6] George C Bush. The embeddability of a semigroup—conditions common to mal'cev and lambek. *Transactions of the American Mathematical Society*, 157:437–448, 1971. 2

[7] Nathan C Carter. *Visual group theory*. MAA Press, 2009. 4