INVESTIGATING THE RELATIONSHIP BETWEEN GROUP-EMBEDDABLE MONOIDS AND BIMORPHISM MONOIDS OF INFINITE GRAPHS

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Funded through the Laidlaw Scholarship Programme in Research and Leadership

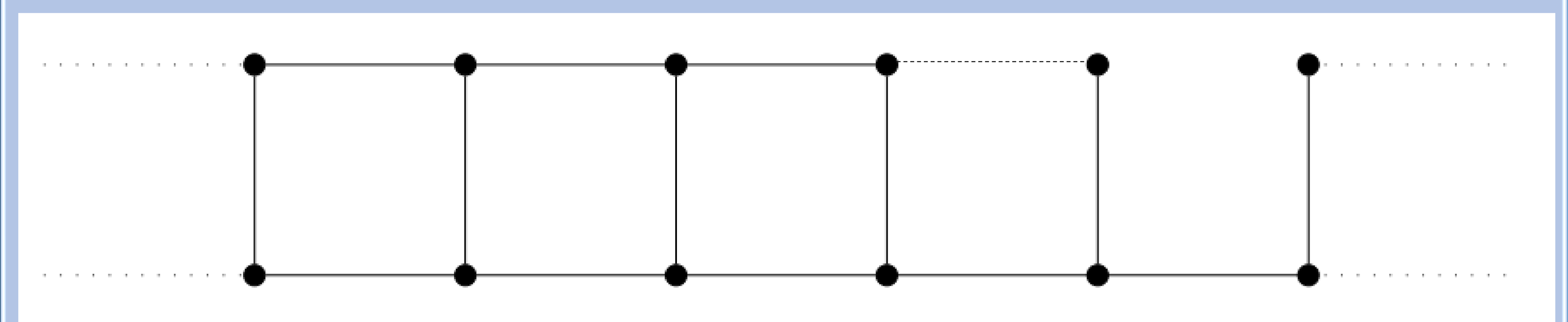
Definitions

- A graph G is an ordered pair (V, E) where V is a set of vertices and $E \subseteq [V]^2$ is a set of edges. If two vertices share an edge, they are said to be adjacent.
- A monoid M is a set equipped with an associative binary operation $\bullet : M \times M \longrightarrow M$ and an identity element e such that for all $x \in M$, $x \bullet e = e \bullet x = x$
- We say a monoid $(M, \bullet)$ can be embedded in a group $(\mathcal{G}, \star)$ if there exists an injective function, $f : M \longrightarrow \mathcal{G}$ such that for all $a, b \in M$, $(a \bullet b)f = af \star bf$.
- A graph bimorphism is a permutation of vertices such that all adjacent vertices are mapped to adjacent vertices. A bimorphism that preserves non-adjacencies is an automorphism. Every bimorphism of a finite graph is an automorphism.

The bimorphism monoid of a graph

The bimorphisms of a graph form a monoid, and because this monoid is made up of permutations of vertices, it can be embedded in the permutation group on its set of vertices, $\mathrm{Sym}(V)$. Therefore, every graph's bimorphism monoid gives an example of a group-embeddable monoid. It is currently unknown whether every group-embeddable monoid arises as the bimorphism monoid of a graph, but if that were the case, it would provide a generalization to a famous theorem by Frucht, which states that every finite group arises as the automorphism group of a finite undirected graph.

Figure 1: A graph with a familiar bimorphism monoid



The dashed edge indicates where an edge relation is added when each vertex is mapped to the one immediately to its left.

The graph in Figure 1 is a graph my supervisor constructed to have a very specific bimorphism monoid. The graph has no automorphisms apart from the identity, but if we consider bimorphisms of the graph, we find that the bimorphism monoid is the non-negative integers under addition. Consider a translation that moves each vertex to the left by one. Adjacent vertices are mapped to adjacent vertices, but we find ourselves adding a relation each time. My supervisor proved that the bimorphisms generated by repeated applications of the above translation were the only non-identity bimorphisms of the graph, which makes the bimorphism monoid the free commutative monoid on one generator. This monoid is the same thing as the non-negative integers under addition

In my research, I sought to construct a graph whose bimorphism monoid was the free commutative monoid on two generators. This is really just the set of all ordered pairs (n, m) where $n, m \in \mathbb{N}_0$ under addition of ordered pairs. I ended up with the graph in Figure 2. The only non-identity bimorphisms on this graph are bimorphisms that move the graph up or to the right.

I was able to find a possible construction for a graph whose bimorphism monoid is the free commutative monoid on an arbitrary number of generators (Figure 3), but unfortunately the method used to prove the one and two generator cases is unsuitable for the general case. I'm hopeful that the general construction will see a proof someday, and perhaps shed some light on the nature of group-embeddable monoids as a whole.

Figure 2: (Right)
A graph with bimorphism monoid isomorphic to the free commutative monoid on two generators.

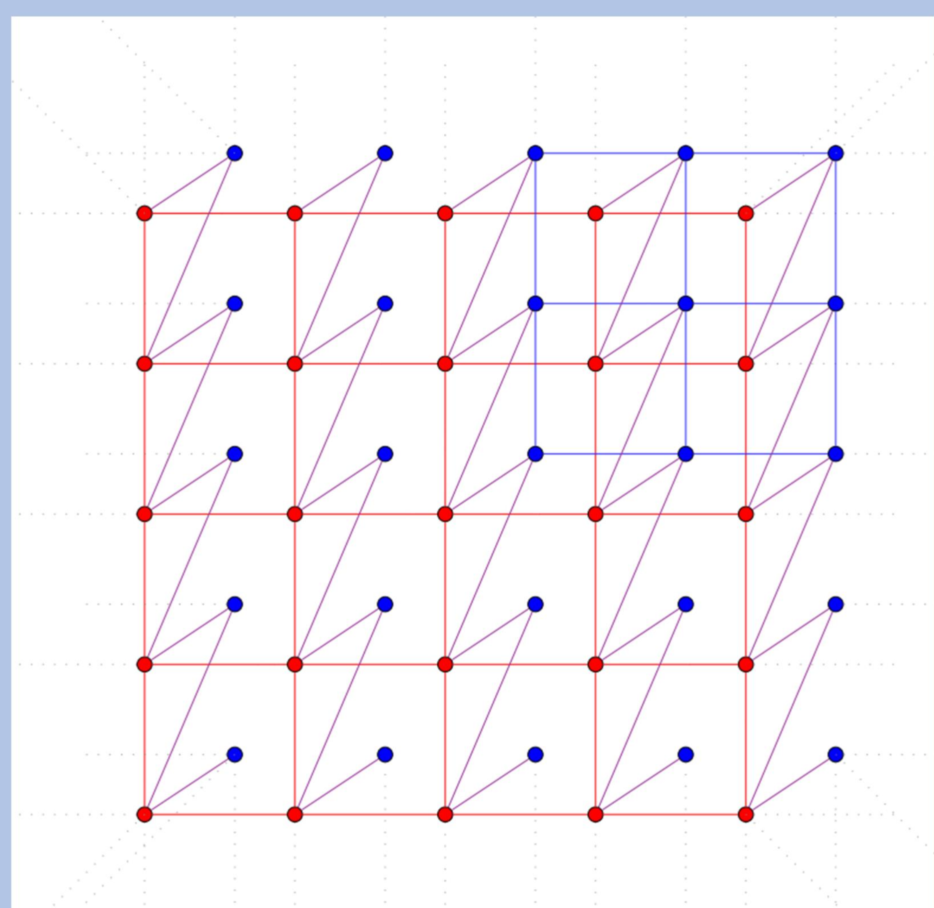


Figure 3: (Below)
A possible construction for a graph with bimorphism monoid isomorphic to the free commutative monoid on n generators.

Let Γ be a graph with vertex set $V(\Gamma) = \mathbb{Z}^n \times \{0, 1\}$, and edge relations given by:

- $(v_1, v_2, \ldots, v_n, 0) \sim (w_1, w_2, \ldots, w_n, 0)$ if and only if
$$\sum_{i=1}^{n} |\, v_i - w_i \,| = 1$$
- $(v_1, v_2, \ldots, v_n, 1) \sim (w_1, w_2, \ldots, w_n, 1)$ if and only if $v_1, w_1, ..., v_n, w_n \geq 0$ and
$$\sum_{i=1}^{n} |\, v_i - w_i \,| = 1$$
- $(v_1, v_2, \ldots, v_n, 0) \sim (w_1, w_2, \ldots, w_n, 1)$ if and only if $v_n = w_n$ and if there exists an i such that $v_i \neq w_i$, then we require that $w_i - v_i = i$, and $v_j = w_j$ for all $j \neq i$.

Acknowledgements

This project would not have been possible without the generous financial support of Lord Laidlaw, and the support and flexibility of the Laidlaw Team in St Andrews. To my supervisor, Dr. Tom Coleman, thank you for the much-needed direction, proof reading, and encouragement. I could not have asked for a better first research experience.