Topological Origami

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Introduction

Rigidity theory is the study of how idealised bar joint frameworks move. Picture the black square in Figure 1, and imagine pushing the top edge so it deforms to form the red rhombus. Notice that, in moving the square, all of the edge lengths have stayed the same – this motion is called a flex. However, observe that the non-edge (diagonal) lengths *have* changed. Flexes where a non-edge length changes are known as non-trivial. If a framework admits a non-trivial flex, we say it is flexible. Otherwise, it is rigid.

For my research project, I have been studying the behaviour of screw-periodic frameworks (see Figure 2) and developing some code to explore how we can generate rigid ones. This piece of research falls under the umbrella of 'topological origami' since we may think of a rigid triangulation modelled on the torus as behaving like a crinkled piece of paper. McInerney et al. present a duality between stresses and twists in [3] that shows such triangulations have two folding motions, and my research looks to supplement this by exploring how these frameworks are generated.

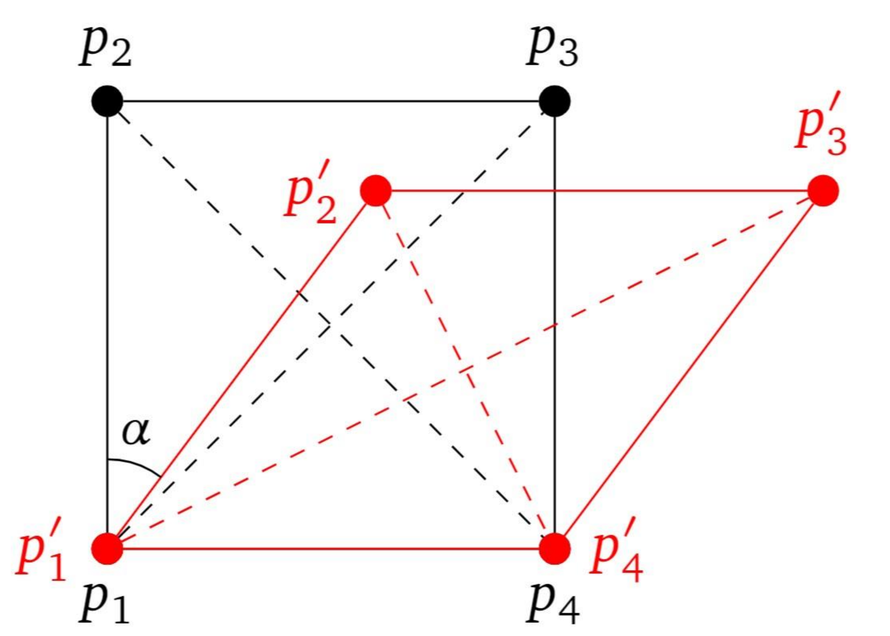

Figure 1: A square framework being deformed into a rhombus.

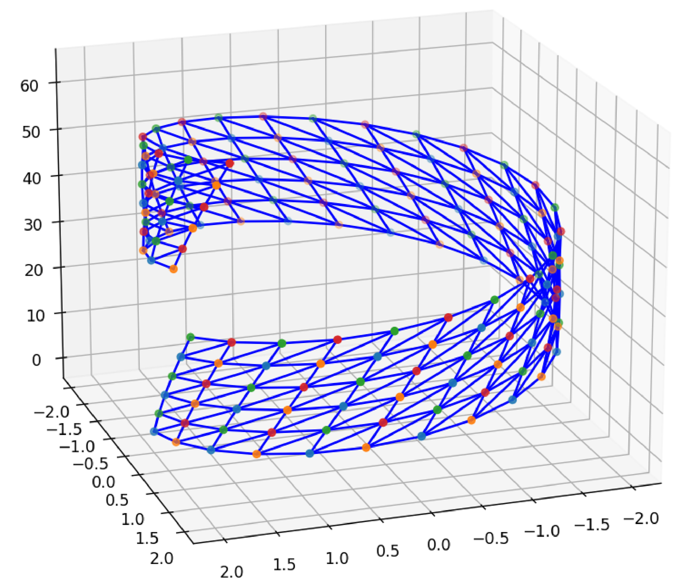

Figure 2: An example of a finite section of an infinite screw-periodic framework on four vertices. This was generated in Python using [1].

Screw-periodic Frameworks

Before we define a screw-periodic framework, we must first define a screw motion as a transformation that is made up of a rotation and a translation down the rotation axis. Then given two screw motions, T_1 and T_2, with rotations that share the same axis, we can (heuristically) define a screw-periodic framework by taking a framework in three dimensions, and applying $T_1^a T_2^b$ to each vertex, for each $(a, b) \in \mathbb{Z}^2$. We can think of the output of this process as an infinite triangulated sheet that has been wrapped around an axis. A finite section of this is demonstrated in Figure 2.

With this definition in mind, the theorem that I proved as part of my research project is:

If a triangulation of the torus is infinitesimally rigid, then the framework created upon performing a vertex split on that triangulation is also infinitesimally rigid with stress space of dimension two.

To understand this further, we must understand what a 'vertex split' is …

Vertex Splits

Take the graph on the right of Figure 3 and imagine contracting the red edge so that the red vertices join together, the two green edges become one, and the two blue edges become one. This would give the graph on the left. Now imagine doing that operation in reverse. This is how we can think of a vertex split - the idea of pulling one vertex apart to form two.

This idea of vertex splitting was first introduced by Whiteley in [4], where he goes on to present a proof of a theorem similar to our theorem above, except in the finite setting. It is this paper from which we draw a lot of inspiration and techniques for our rigidity theoretical reasoning.

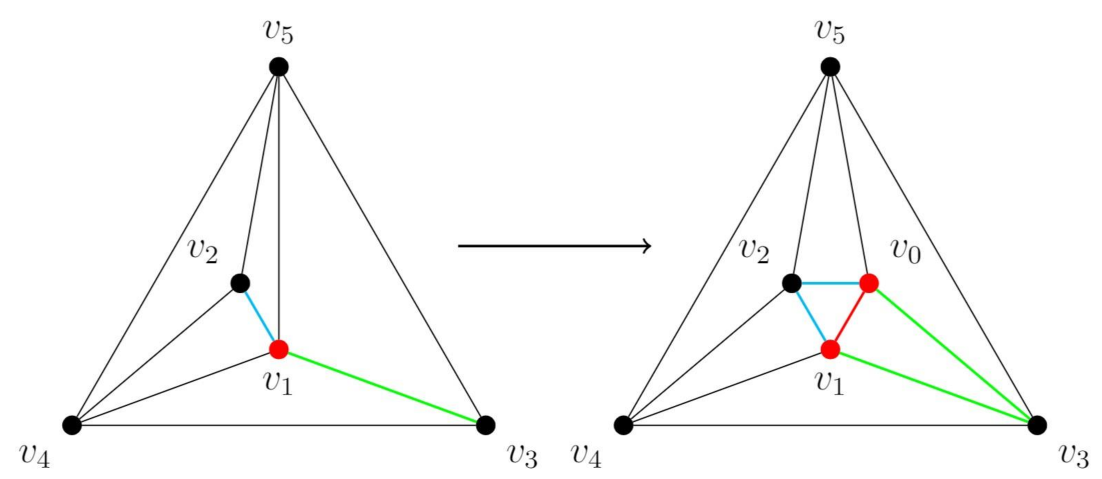

Figure 3: A vertex split on two edges in 3 dimensions.

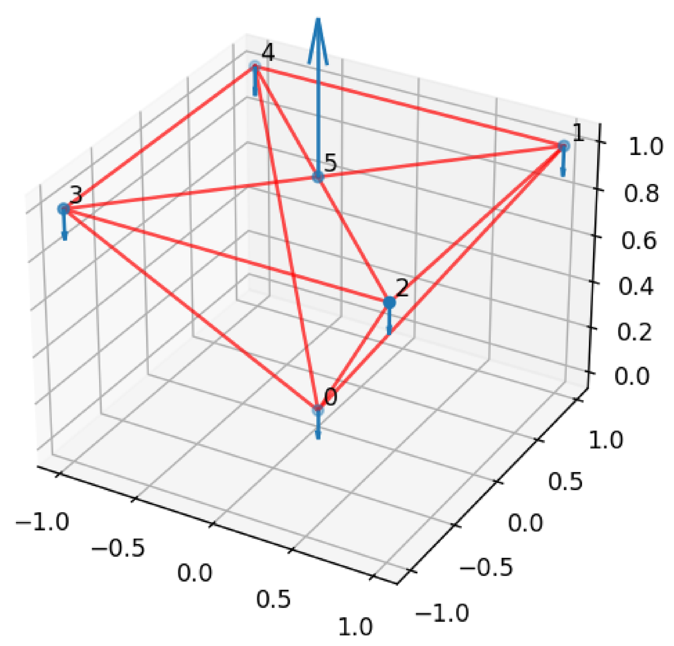

Figure 4: An example of an infinitesimal flex on a framework in three dimensions. Adapted from Figure 1 in [2], this graphic was generated in Python using [1].

Framework Modelling

The theorem we stated above is in fact an intermediate result that will be used to prove the larger proposition that any screw-periodic framework modelled on a triangulation of a torus is generically screw-periodically rigid. The second half of this proof will consist of a combinatorial argument that confirms every screw-periodic graph can be formed by a series of vertex splits on a finite base set of graphs. In order to do this in the future, I developed the Python Package `Rigidity` [1].

`Rigidity` is a linear algebra-type package that allows for rapid computation of framework operations such as generating the rigidity matrix, calculating non-trivial infinitesimal flexes, and performing vertex splits. Further to this, it allows the user to generate images of frameworks and their flexes, as in Figure 2 and 4, in the spirit that seeing things makes them easier to understand. I certainly believe that, through developing this package and generating many examples, I have gained a deeper appreciation and understanding for the underlying maths.

Summary

At the outset of my project, I presented a proposition in relation to the nature of rigidity in screw-periodic frameworks. I claimed that any screw-periodic graph modelled on a triangulation of a torus is generically screw-periodically rigid. To prove this, I first proved an intermediate result that states vertex splitting preserves infinitesimal rigidity on screw-periodic frameworks. Having done this, I am left to show that every screw-periodic graph modelled on the torus is generated by a series of vertex splits from a finite collection of base cases. This second part of the proof will require a lot of cumbersome computation, and hence I developed a Python package [1] that, in the future, will be used to perform these computations and complete the proof. (This Python package also contains the code used to generate the Figures 2 and 4).

References

[1] Joseph Edwards and Louis Theran. *Rigidity: A Package for Studying Rigidity Theory.* Aug. 2021. URL: https://github.com/theran/laidlaw-origami
[2] Herman Gluck. "Almost all simply connected closed surfaces are rigid". In: *Geometric Topology*. Ed. by Leslie Curtis Glaser and Thomas Benjamin Rushing. Berlin, Heidelberg: Springer Berlin Heidelberg, 1975, pp. 225–239.isbn:978-3-540-37412-1.
[3] James McInerney et al. "Hidden symmetries generate rigid folding mechanisms in periodic origami". Proceedings of the National Academy of Sciences 117.48 (Nov. 2020), pp. 30252–30259.issn: 1091-6490. doi: 10.1073/pnas.2005089117.url:http://dx.doi.org/10.1073/pnas.2005089117.
[4] Walter Whiteley. "Vertex Splitting in Isostatic Frameworks". In:StructuralTopology16 (Jan. 1990).

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