Understanding Pseudo-Similarity in Graphs:
A Path to Proving the Reconstruction Conjecture

Frankie Gillis
School of Mathematics and Statistics

1. Introduction: Graphs, Symmetries and Partial Symmetries

A *graph* is a mathematical object made up of a set of vertices and a set of edges connecting them. Graphs have fundamental real-world and theoretical applications, such as modelling friendships on social networks, or constructing neural networks in artificial intelligence. Of key interest in pure mathematics are the *symmetries* of graphs, as these symmetries provide a systematic way to study, understand, and classify graphs.

Symmetry is formalised by an *automorphism*, which is a relabeling of the vertices that preserves the positions of the graph's edges. We call two vertices *similar* if there is an automorphism mapping one onto the other – intuitively, similar vertices can be exchanged without affecting any connections in the graph. Pairs of vertices can act like similar vertices without being genuinely similar. These vertices are called *pseudo-similar*, and are poorly understood.

Interest in the study of *partial* automorphisms has been sparked by pseudo-similarity. These generalise the idea of symmetries to 'partial' symmetries between smaller parts of graphs. The goal of my research would be, broadly, to expand the theory of partial automorphisms to better understand pseudo-similarity. Graph theory lies at the intersection of many areas of pure and applied mathematics: algebra, combinatorics, topology, and computer science. As an aspiring researcher in pure mathematics, this combination is especially motivating.

2. Connection to the Reconstruction Conjecture

A key motivation for studying pseudo-similarity is a long-standing conjecture in graph theory: the *reconstruction conjecture*. It claims that any graph can be built uniquely given only information about the smaller graphs contained within it. A proof exists for graphs without pseudo-similar vertices, so understanding pseudo-similarity through partial automorphisms could be a key avenue towards a complete proof. Thus, my research could impact a progressing field of pure mathematics.

3. Research Methodology

My research will take place at St Andrews within the School of Mathematics and Statistics. My supervisor will define the scope of my research, which problems to tackle, and recommend resources. My methodology will be iterative: identifying patterns, formulating conjectures, and proving them. This predominantly takes place at the blackboard, with the aid of computational tools such as Python's NetworkX package.

4. Key Goals

Key achievements of my research will be answering:

- For each integer k, what is the smallest graph having k mutually-pseudo-similar vertices?
- A method is known to construct graphs with a pair of vertices which are either similar or pseudo-similar; can this method be modified to produce pseudo-similar vertices more successfully?

In preparation for this project, I solved a similar, smaller-scale problem: constructing the smallest possible graph with no symmetries. The key to my success was discovering a link between the size of a graph and its automorphism group. Hence, by advancing the theory of partial automorphisms and pseudo-similarity, my research will contribute to a newly-progressing field of pure mathematics and aid in solving a central conjecture.