

Understanding Pseudo-similarity in Graphs: a Path to Proving the Reconstruction Conjecture

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1 Introduction

Two vertices u and v in a graph Γ are called *pseudo-similar* if the vertex-induced subgraphs $\Gamma - u$ and $\Gamma - v$ are isomorphic, but no automorphism of Γ maps u to v . Pseudo-similarity was discovered during attempts to prove the long-standing *Graph Reconstruction Conjecture*, of Kelly and Ulam [Kel57, Ula60]. Given a graph Γ , we define the *deck* of Γ , denoted $\text{Deck}(\Gamma)$, to be the multiset of isomorphism classes of the graphs $\Gamma - u$ for each $u \in V\Gamma$. The Reconstruction Conjecture claims that every finite graph of order at least 3 is uniquely reconstructible from its deck; that is, given any two finite graphs Γ and Γ' on at least 3 vertices, $\text{Deck}(\Gamma) \cong \text{Deck}(\Gamma')$ if and only if $\Gamma \cong \Gamma'$. The Reconstruction Conjecture has resisted traditional algebraic methods for a considerable amount of time [NW78]. In particular, it is claimed in [HP66, Sto81] that a ‘proof’ of the Reconstruction Conjecture emerged in the 1960s which relied heavily on the non-existence of pseudo-similar vertices. This suggests that a valid proof of the Reconstruction Conjecture may be found through better understanding finite graphs which contain pseudo-similar vertices.

Alongside its study in connection with the Reconstruction Conjecture, pseudo-similarity has attracted interest for study in its own right, for example in [GK82, GK83, Lau97]. In particular, finding the size of the largest possible set of pseudo-similar vertices that a graph can contain has been the subject of intense research efforts [KP75, KJSS81], and the problem of determining the orders of the smallest graphs Γ_k each containing a set of mutually pseudo-similar vertices is still an open problem.

Both of these problems are closely related to the study of isomorphisms between induced subgraphs of a graph Γ . In [JJSS21], it is argued that the study of the *inverse monoid of partial automorphisms* of a graph Γ may be key to breakthroughs in problems like the Reconstruction Conjecture or understanding pseudo-similar vertices. A *partial automorphism* of a graph Γ is an isomorphism between its vertex-induced subgraphs, and the partial automorphisms of a graph Γ form an inverse monoid under composition of partial maps, denoted $\text{PAut}(\Gamma)$. This monoid may be viewed as a generalisation of the automorphism group of Γ , as $\text{PAut}(\Gamma)$ contains $\text{Aut}(\Gamma)$ as a subgroup. $\text{PAut}(\Gamma)$ is a rich source of additional information about the algebraic structure of Γ . For example, while many non-isomorphic graphs may have the same automorphism group (even more so, in [ER63] it is proven that almost all graphs have a trivial automorphism group), for any two finite graphs Γ and Γ' , we have $\text{PAut}(\Gamma) \cong \text{PAut}(\Gamma')$ if and only if $\Gamma \cong \Gamma'$ or $\bar{\Gamma} \cong \bar{\Gamma}'$, where $\bar{\Gamma}$ denotes the complement of Γ .

In this project, we study how the algebraic theory of inverse monoids of partial automorphisms of graphs began in [JJSS21] can be applied to understanding pseudo-similar vertices in graphs, with a particular emphasis on graphs with sets of mutually pseudo-similar vertices. These results are used to improve the best known upper bound on the size of a set of mutually pseudo-similar vertices in a graph Γ . Finally, we consider digraphs which contain pseudo-similar vertices. In general, it is easier to construct digraphs with pseudo-similar vertices than graphs. The falsity of the Reconstruction Conjecture for digraphs seems to support the connection between pseudo-similarity and graph reconstruction; indeed, the existence of certain families of digraphs with large sets of pseudo-similar vertices is key in the construction of infinite families of counterexamples to the digraph Reconstruction Conjecture [Sto77, Sto81]. We will find that the subtleties which exist for digraphs cause many of the results we prove for graphs to fail for digraphs.

1.1 Prerequisites from Graph Theory

A *graph* Γ is an ordered pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is the *vertex set* and $E\Gamma \subseteq [V\Gamma]^2$ is the *edge set*, which is a collection of 2-element subsets of $V\Gamma$. A *digraph* Γ is an ordered pair $(V\Gamma, E\Gamma)$ where $V\Gamma$ is the *vertex set* and $E\Gamma \subseteq V\Gamma \times V\Gamma$ is the *edge set*, which is a collection of ordered pairs of elements of $V\Gamma$. Digraphs may have loops, but have no multiple edges. All graphs and digraphs considered are finite. For more detail on general graph theory, see [Die24].

If Γ is a graph, and the vertices u and v are adjacent, then we write $u \sim v$. The *neighbourhood* of the vertex u in Γ is denoted by $N_\Gamma(u)$. The *degree* of the vertex u in Γ is written as $\deg_\Gamma(u)$. The *degree sequence* of Γ is the list of all the degrees of the vertices in Γ , with multiplicity, written in increasing order.

If Γ is a digraph, we write $u \rightarrow v$ to mean there is a directed edge from the vertex u to the vertex v . The *in-neighbourhood* of u in Γ is written as $N_{\Gamma \rightarrow}(u)$. Similarly, the *out-neighbourhood* of u in Γ is written as $N_{\Gamma \leftarrow}(u)$. If Γ is a digraph, we write $N_\Gamma(u)$ to mean $N_{\Gamma \rightarrow}(u) \cup N_{\Gamma \leftarrow}(u)$. The *extended neighbourhood* of u in Γ is the set $N'_\Gamma(u) = N_\Gamma(u) \cup \{u\}$.

An *isomorphism* φ between two graphs Γ, Γ' is a bijection

$$\varphi : V\Gamma \rightarrow V\Gamma'$$

which preserves both edges and non-edges. We denote isomorphism by $\Gamma \cong \Gamma'$. An *automorphism* is an isomorphism from a graph to itself. The group of automorphisms of Γ is denoted by $\text{Aut}(\Gamma)$.

If Γ is a graph or a digraph, then the *induced subgraph* of Γ with vertex set $A \subseteq V\Gamma$ is denoted by $\Gamma[A]$. Given an isomorphism

$$\varphi : \Gamma[A] \rightarrow \Gamma[B]$$

between induced subgraphs of Γ , we abuse notation and write

$$\varphi : A \rightarrow B$$

where this will not cause confusion.

If $u \in V\Gamma$, the induced subgraph $\Gamma[V\Gamma \setminus \{u\}]$ is denoted by $\Gamma - u$. If $u, v \in V\Gamma$, the induced subgraph $\Gamma[V\Gamma \setminus \{u, v\}]$ is denoted by $\Gamma - u - v$.

We define the following two equivalence relations on $V\Gamma$.

1. *Similarity*: If $u, v \in V\Gamma$, then u and v are called *similar* if there is a $\varphi \in \text{Aut}(\Gamma)$ such that $u\varphi = v$. This is denoted by $u \sim_S v$.
2. *Removal-similarity*: If $u, v \in V\Gamma$, then u and v are called *removal-similar* if $\Gamma - u \cong \Gamma - v$. This is denoted $u \sim_R v$.

Definition 1.1 (Pseudo-similarity). Let Γ be a graph (or a digraph), and let $u, v \in V\Gamma$. Then u and v are called *pseudo-similar* if they are removal-similar but not similar. That is, if $u \sim_R v$ but not $u \sim_S v$.

While similarity and removal-similarity are both equivalence relations, pseudo-similarity is *not* an equivalence relation. It is irreflexive (since a vertex is always similar to itself, by the identity

automorphism), and is not, in general, transitive. Therefore, in order to study sets of vertices which are mutually pseudo-similar, the following definition is made.

Definition 1.2 (Mutually pseudo-similar vertices). Let Γ be a graph (or a digraph), and let $A = \{u_1, \dots, u_k\} \subseteq V\Gamma$, with $k \geq 2$. The vertices in A are called *mutually* pseudo-similar if they are all pairwise pseudo-similar.

In [Lau97], it is observed that many of the most interesting questions about pseudo-similarity in graphs are concerned with sets of mutually pseudo-similar vertices, and these questions have attracted the attention of many authors. Mutually pseudo-similar vertices also appear to be the most enigmatic: while existence of pairs of pseudo-similar vertices has an explanation in terms of a kind of truncation of cyclic symmetry—see [GK82]—no such explanation exists for larger sets of mutually pseudo-similar vertices. Much of the work in this essay is concerned with finding necessary and sufficient conditions for sets of mutually pseudo-similar vertices to exist. In particular, we consider the following question, first asked in [KJSS81] and most recently in [JJSS21].

Problem 1.3. *For each k , what is the order of the smallest graph Γ_k containing a set of k mutually pseudo-similar vertices?*

1.2 Inverse Monoids of Partial Graph Automorphisms

A *partial automorphism* of a graph Γ is an isomorphism between two of its vertex-induced subgraphs. The partial automorphisms of Γ form an *inverse monoid* under the operation of composition of partial maps. This monoid is called the *inverse monoid of partial automorphisms* of Γ , or the *partial automorphism monoid* of Γ , and is denoted $\text{PAut}(\Gamma)$. We denote the domain of a partial automorphism φ by $\text{dom}(\varphi)$, and the image by $\text{im}(\varphi)$. The *rank* of a partial automorphism is the size of its domain.

We are interested in two aspects of the algebraic structure of such monoids; namely, Green's relations and the natural partial order, which we shall review. For a good source on general semigroup theory, see [How95]. The theory we review here is developed in more detail in [JJSS21]. We use the cycle-and-path notation from [JJSS21] to represent partial permutations, however we write mappings on the *right* of their arguments, and compose maps from *left to right*. This, in particular, has the effect of swapping the roles of the \mathcal{L} and \mathcal{R} -relations. We denote composition of partial maps by \circ .

For a monoid M , Green's relations are a collection of equivalence relations defined on M which describe divisibility in M . These are \mathcal{L} , \mathcal{R} , \mathcal{H} , \mathcal{D} , and \mathcal{J} ; the final two of which coincide in the case M is finite. For $a, b \in M$, the \mathcal{L} -relation is defined by

$$a\mathcal{L}b \text{ if there exist } x, y \in M \text{ such that } xa = b \text{ and } yb = a.$$

Dually, the \mathcal{R} -relation is defined by

$$a\mathcal{R}b \text{ if there exist } x, y \in M \text{ such that } ax = b \text{ and } by = a.$$

The \mathcal{H} -relation is defined by $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$. That is, $a\mathcal{H}b$ if and only if $a\mathcal{L}b$ and $a\mathcal{R}b$. The \mathcal{D} -relation is the largest equivalence relation containing both the \mathcal{L} and \mathcal{R} -relations. For $a, b \in M$,

$$a\mathcal{D}b \text{ if there exists } x \in M \text{ such that } a\mathcal{L}x\mathcal{R}b.$$

The final relation, \mathcal{J} is equal to \mathcal{D} in the case when M is finite (Chapter 2, [How95]). We are interested in studying Green's relations on the partial automorphism monoid of a graph. When Γ is a finite graph, $\text{PAut}(\Gamma)$ is also finite, therefore we will not need to consider the \mathcal{J} -relation.

To study pseudo-similarity in graphs, we need to understand which graph-theoretic properties are described by Green's relations on $\text{PAut}(\Gamma)$. The following proposition is from [JJSS21].

Proposition 1.4. *Let Γ be a graph, and $\varphi, \psi \in \text{PAut}(\Gamma)$. Then*

1. $\varphi \mathcal{L} \psi$ if and only if $\text{im}(\varphi) = \text{im}(\psi)$.
2. $\varphi \mathcal{R} \psi$ if and only if $\text{dom}(\varphi) = \text{dom}(\psi)$.
3. $\varphi \mathcal{H} \psi$ if and only if $\text{dom}(\varphi) = \text{dom}(\psi)$ and $\text{im}(\varphi) = \text{im}(\psi)$.
4. $\varphi \mathcal{D} \psi$ if and only if $\text{dom}(\varphi) \cong \text{dom}(\psi)$.

Proof. Let Γ be a graph, and let $\varphi, \psi \in \text{PAut}(\Gamma)$.

1. Suppose $\varphi \mathcal{L} \psi$. Then, there exist $\alpha, \beta \in \text{PAut}(\Gamma)$ such that $\alpha \circ \varphi = \psi$ and $\beta \circ \psi$. Since $\text{im}(\alpha \circ \varphi) \subseteq \text{im}(\varphi)$, it follows that $\text{im}(\psi) \subseteq \text{im}(\varphi)$. The other containment comes from the symmetric argument considering $\beta \circ \psi$. Now suppose $\text{im}(\varphi) = \text{im}(\psi)$. Then, $(\psi \circ \varphi^{-1}) \circ \varphi = \psi \circ (\varphi^{-1} \circ \varphi) = \psi \circ \text{id}_{\text{im}(\varphi)} = \psi$. The other product follows from swapping the roles of φ and ψ . Hence $\varphi \mathcal{L} \psi$.
2. This argument is dual to the argument in (1).
3. Since $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$, it follows that $\varphi \mathcal{H} \psi$ if and only if $\varphi \mathcal{L} \psi$ and $\varphi \mathcal{R} \psi$. Therefore, $\varphi \mathcal{H} \psi$ if and only if $\text{im}(\varphi) = \text{im}(\psi)$ and $\text{dom}(\varphi) = \text{dom}(\psi)$.
4. $\varphi \mathcal{D} \psi$ if and only if there exists $\alpha \in \text{PAut}(\Gamma)$ such that $\varphi \mathcal{L} \alpha \mathcal{R} \psi$. The first relation holds if and only if $\text{im}(\varphi) = \text{im}(\alpha)$, and the latter relation if and only if $\text{dom}(\alpha) = \text{dom}(\psi)$. Hence, $\varphi \mathcal{D} \psi$ if and only if there exists an isomorphism $\alpha : \text{dom}(\psi) \rightarrow \text{im}(\varphi)$. This is equivalent to $\text{dom}(\psi) \cong \text{im}(\varphi)$, or equivalently, $\text{dom}(\psi) \cong \text{dom}(\varphi)$.

□

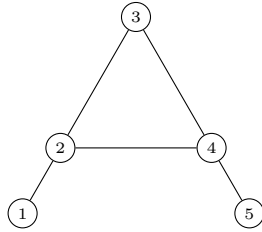
Remark. Proposition 1.4 is also true if we replace Γ with a digraph. The proof is the same.

Now we define the second aspect of the structure of $\text{PAut}(\Gamma)$ we will need in order to study pseudo-similarity: the *natural partial order* on $\text{PAut}(\Gamma)$. Given any induced subgraph $\Gamma[A]$ of Γ , the identity map on $\Gamma[A]$ is a partial automorphism, and is denoted $\text{id}_{\Gamma[A]}$. The identity maps are *idempotents*, satisfying $(\text{id}_{\Gamma[A]})^2 = \text{id}_{\Gamma[A]}$, and are the only such idempotents of $\text{PAut}(\Gamma)$. These idempotents can be used to define a partial order on $\text{PAut}(\Gamma)$: given $\varphi, \psi \in \text{PAut}(\Gamma)$, we define $\varphi \leq \psi$ if and only if there exists an idempotent e such that $\varphi = e \circ \psi$. This is the restriction ordering on $\text{PAut}(\Gamma)$: for any $\varphi, \psi \in \text{PAut}(\Gamma)$, the relation $\varphi \leq \psi$ is true if and only if φ is obtained by restricting the domain of ψ [JJSS21]. Where this is the case, ψ will be called an *extension* of φ , and φ is a partial automorphism *induced* by ψ . An element $\varphi \in \text{PAut}(\Gamma)$ is called *maximal* in the partial order if there is no element $\psi \in \text{PAut}(\Gamma)$ such that $\varphi \leq \psi$. That is, a partial automorphism is maximal if it cannot be extended to a partial automorphism with a larger domain.

2 Main Results

2.1 When does a graph contain a pair of pseudo-similar vertices?

It may first seem the case that vertices u, v are pseudo-similar in the graph Γ if there exists a partial automorphism $\varphi : \Gamma - u \rightarrow \Gamma - v$ which cannot be extended to an automorphism of Γ mapping u to v . This is, however, not sufficient, as shown in [Example 2.1](#).



Γ

Figure 1: The graph Γ .

Example 2.1. Consider the graph Γ in [Figure 1](#). The map

$$\varphi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & - \end{pmatrix}$$

is a partial automorphism of Γ of rank $|V\Gamma| - 1$ which maps the graph $\Gamma - 5$ to the graph $\Gamma - 1$. φ cannot be extended to an automorphism of Γ , as the only permutation which extends φ is

$$\varphi' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$$

which maps the edge $\{1, 2\}$ onto the non-edge $\{5, 3\}$. However, the vertices 1 and 2 are not pseudo-similar in Γ . The map

$$\psi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix}$$

is an automorphism of Γ mapping 5 to 1, therefore $1 \sim_S 5$ in Γ .

The correct condition is stronger, and involves Green's \mathcal{H} -relation in $\text{PAut}(\Gamma)$.

Theorem 2.2 (Pseudo-similarity). *Let Γ be a graph with removal-similar vertices $u, v \in V\Gamma$. Then the following are equivalent:*

1. *The vertices u and v are pseudo-similar in Γ .*

2. The \mathcal{H} -class H_φ of any partial automorphism $\varphi : \Gamma - u \rightarrow \Gamma - v$ has the property that every $\psi \in H_\varphi$ is maximal in the natural partial order on $\text{PAut}(\Gamma)$.

Proof. Let Γ , u and v be as given in the statement of the theorem. Suppose first that u and v are pseudo-similar in Γ . By assumption, $u \sim_R v$, so there exists a partial automorphism of Γ

$$\varphi : \Gamma - u \rightarrow \Gamma - v.$$

Suppose, for a contradiction, that there exists $\psi \in H_\varphi$ which is not maximal in the natural partial order on $\text{PAut}(\Gamma)$. By [Proposition 1.4](#), $\text{dom}(\psi) = \Gamma - u$, and $\text{im}(\psi) = \Gamma - v$, and so

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Therefore, any extension ψ' of ψ must be of the form $\psi \vee (u v]$, and so maps u to v . Since ψ is not maximal in the natural partial order on $\text{PAut}(\Gamma)$, it follows that $\psi' \in \text{PAut}(\Gamma)$. Therefore, there is an automorphism ψ' of Γ which maps u to v . It follows that $u \sim_S v$ in Γ . This is a contradiction. Therefore, we must conclude that every $\psi \in H_\varphi$ is maximal in the partial order on $\text{PAut}(\Gamma)$.

For the backward direction, let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism. Such a φ exists by the assumption that $u \sim_R v$. Suppose that every $\psi \in H_\varphi$ is maximal in the partial order on $\text{PAut}(\Gamma)$. We claim that no automorphism of Γ maps u to v . Suppose for a contradiction that $\theta \in \text{Aut}(\Gamma)$ is such that $u\theta = v$. Then, θ induces a partial automorphism

$$\theta' : \Gamma - u \rightarrow \Gamma - v.$$

By [Proposition 1.4](#), $\theta' \in H_\varphi$. However, $\theta' = \text{id}_{\Gamma-u} \circ \theta$, and so $\theta' \leq \theta$ in the partial order on $\text{PAut}(\Gamma)$. This contradicts the assumed maximality of every $\psi \in H_\varphi$. Therefore, if every $\psi \in H_\varphi$ is maximal in the natural partial order on $\text{PAut}(\Gamma)$, then u and v are not similar. It follows that they are pseudo-similar. \square

2.2 When does a graph contain a set of mutually pseudo-similar vertices?

Following [Theorem 2.2](#), it seems appropriate to seek a characterisation of sets of mutually pseudo-similar vertices in terms of a graph's inverse monoid of partial automorphisms. Here we encounter the problem that pseudo-similarity is not an equivalence relation: it is irreflexive, and importantly, is not transitive. This is illustrated in examples [2.3](#) and [2.4](#). In contrast, Green's relations are all equivalence relations, which presents a difficulty in obtaining the desired characterisation. Given a \mathcal{D} -class D of rank $|V\Gamma| - 1$ elements in $\text{PAut}(\Gamma)$, there may exist partial automorphisms in D (with their domain not equal to their image) which can be extended to automorphisms of Γ , and still, there may exist other partial automorphisms in D which cannot be extended to automorphisms of Γ . Two examples of this are given, one for digraphs and one for graphs; [Figure 2](#) and [Figure 3](#) respectively.

Example 2.3. Consider the digraph Γ in [Figure 2](#). The vertex 1 is pseudo-similar to the vertex 5, and the vertex 5 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is not transitive in digraphs.

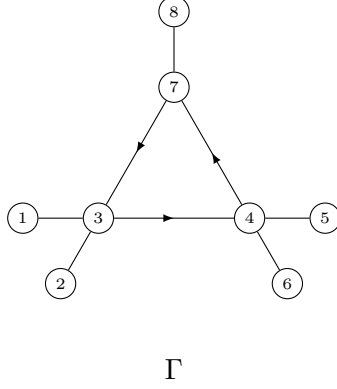


Figure 2: The digraph Γ .

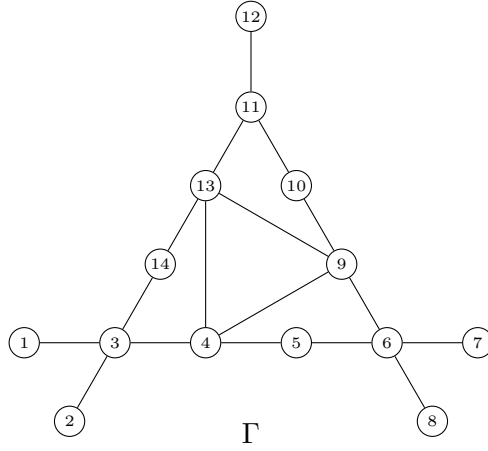


Figure 3: The graph Γ .

Example 2.4. Consider the graph Γ in [Figure 3](#). The vertex 1 is pseudo-similar to the vertex 7, and the vertex 7 is pseudo-similar to the vertex 2. However, the vertices 1 and 2 are similar, and therefore are not pseudo-similar, showing that pseudo-similarity is also not transitive in graphs.

For a set $A \subseteq V\Gamma$ to be mutually pseudo-similar, we must require that the conditions in [Theorem 2.2](#) hold *pairwise* between the vertices in A . First, for concision, the following definition is made.

Definition 2.5. Given a graph Γ , and an \mathcal{H} -class H in $\text{PAut}(\Gamma)$, then we say that H has the *element maximality property* if every $\varphi \in H$ is maximal in the natural partial order on $\text{PAut}(\Gamma)$.

Theorem 2.6 (Mutually pseudo-similar vertices). *Let Γ be a graph, and let $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$. Then the vertices in A are mutually pseudo-similar if and only if*

1. The identity maps $\text{id}_{\Gamma - u_1}, \text{id}_{\Gamma - u_2}, \dots, \text{id}_{\Gamma - u_k}$ are all \mathcal{D} -related.
2. For any pair $u_i, u_j \in A$ of distinct vertices, the \mathcal{H} -class of any partial automorphism $\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$ has the element maximality property.

Proof. First suppose that Γ contains a set $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$ of vertices which are mutually pseudo-similar. Let $u_i, u_j \in A$, where $i \neq j$. By assumption, u_i and u_j are pseudo-similar, so

are both removal-similar and not similar. Since u_i and u_j are removal-similar, $\Gamma - u_i \cong \Gamma - u_j$ and therefore $\text{id}_{\Gamma - u_i} \mathcal{D} \text{id}_{\Gamma - u_j}$ by [Proposition 1.4](#). Moreover, since u_i and u_j are pseudo-similar, [Theorem 2.2](#) implies that the \mathcal{H} -class of any partial automorphism

$$\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$$

has the element maximality property. Since u_i and u_j were chosen arbitrarily, it follows that conditions (1) and (2) hold. This completes the forward direction.

Now suppose that Γ contains a set $A = \{u_1, u_2, \dots, u_k\} \subseteq V\Gamma$ which satisfies conditions (1) and (2). Let $u_i, u_j \in A$, where $u_i \neq u_j$. By the assumption in condition (1), $\text{id}_{\Gamma - u_i} \mathcal{D} \text{id}_{\Gamma - u_j}$. Moreover, by [Proposition 1.4](#),

$$\text{dom}(\text{id}_{\Gamma - u_i}) \cong \text{dom}(\text{id}_{\Gamma - u_j}),$$

so $\Gamma - u_i \cong \Gamma - u_j$. Therefore u_i and u_j are removal-similar.

By (2), the \mathcal{H} -class of any partial automorphism

$$\varphi : \Gamma - u_i \rightarrow \Gamma - u_j$$

has the element maximality property, and so, by [Theorem 2.2](#), u_i and u_j are pseudo-similar. Since u_i and u_j were chosen arbitrarily, the vertices in A are pairwise pseudo-similar, hence are mutually pseudo-similar. □

Remark. Theorems [2.2](#) and [2.6](#) are also true for digraphs. The proofs are identical.

[Theorem 2.6](#) is limited in the sense that it is essentially [Theorem 2.2](#) applied pairwise to the vertices in the set A . If these methods are to be used to provide a resolution to [Question 1.3](#), then a stronger characterisation of mutually pseudo-similar vertices should be found. What is needed is a more complete theory of how Green's relations interact with the natural partial order on an inverse monoid M , in particular in the case when M is the partial automorphism monoid of a graph or a digraph.

2.3 When can a partial automorphism between vertex-deleted subgraphs of a graph Γ be extended to an automorphism of Γ ?

This section considers when a partial automorphism of rank $|V\Gamma| - 1$ in a graph Γ can be extended to an automorphism of Γ .

Lemma 2.7. *Let Γ be a graph and $u, v \in V\Gamma$ be vertices. Let $\varphi : \Gamma - u \rightarrow \Gamma - v$ be a partial automorphism of Γ . Then φ can be extended to an automorphism of Γ if and only if φ induces a partial automorphism*

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Proof. Let Γ , u , v , and φ be as given in the statement of the lemma. Suppose φ induces a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Since $\varphi : \Gamma - u \rightarrow \Gamma - v$, the only extension of φ is the function $\varphi' = \varphi \vee (uv]$ which maps u to v . A permutation of $V\Gamma$ is an automorphism of Γ if and only if it preserves both the edge and non-edge relations on $V\Gamma$; since φ' is an extension of the partial automorphism φ , it follows that φ' is an automorphism of Γ if and only if

$$u \sim w \text{ if and only if } v \sim w\varphi, \text{ for all } w \in V\Gamma.$$

Let $w \in V\Gamma$. Then $u \sim w$ if and only if $w \in N_\Gamma(u)$. Since φ induces the partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

it follows that $w\varphi \in N_\Gamma(v)$ if and only if $w \in N_\Gamma(u)$. Hence, $u \sim w$ if and only if $v \sim w\varphi$. Therefore, φ' is an automorphism of Γ . This completes the forward direction.

For the backward direction, suppose that φ does not induce a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Then, there exists $w \in N_\Gamma(u)$ such that $w\varphi \notin N_\Gamma(v)$. Therefore, w is such that $u \sim w$ but not $v \sim w\varphi'$, and so φ' is not an automorphism of Γ . This completes the backward direction. \square

Theorem 2.8. *Let Γ be a graph with vertices $u, v \in V\Gamma$. Suppose that $\varphi : \Gamma - u \rightarrow \Gamma - v$ is a partial automorphism of Γ , and let A denote the image of $N_\Gamma(u)$ under φ . Then, u and v are similar in Γ if and only if $\Gamma - v$ has an automorphism mapping A to $N_\Gamma(v)$.*

Proof. Let Γ, u, v, φ and A be as given in the statement of the theorem. Suppose first that θ is an automorphism of $\Gamma - v$ such that $A\theta = N_\Gamma(v)$. We will show that $u \sim_S v$ in Γ . Consider the partial automorphism $\psi = \varphi \circ \theta$. Since

$$\varphi : \Gamma - u \rightarrow \Gamma - v,$$

and

$$\theta : \Gamma - v \rightarrow \Gamma - v,$$

it follows that $\text{dom}(\psi) = \Gamma - u$, and $\text{im}(\psi) = \Gamma - v$. Therefore,

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Now, consider the image of $N_\Gamma(u)$ under ψ . From the definition of ψ ,

$$\begin{aligned} (N_\Gamma(u))\psi &= (N_\Gamma(u))(\varphi\theta) \\ &= ((N_\Gamma(u))\varphi)\theta \\ &= A\theta \\ &= N_\Gamma(v). \end{aligned}$$

It follows that ψ induces a partial automorphism

$$\alpha : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and hence, by [Lemma 2.7](#), ψ can be extended to an automorphism of Γ mapping u to v . Therefore, $u \sim_S v$ in Γ .

Now suppose that $u \sim_S v$ in Γ . We will show that $\Gamma - v$ has an automorphism mapping A to $N_\Gamma(v)$. Since u and v are similar in Γ , there exists an automorphism α of Γ such that $u\alpha = v$. Therefore, restricting α induces a partial automorphism

$$\psi : \Gamma - u \rightarrow \Gamma - v.$$

Moreover, it follows, since ψ is induced from by automorphism of Γ , that ψ induces a partial automorphism

$$\beta : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Consider the partial automorphism $\theta = \varphi^{-1}\psi$. Notice that, since

$$\varphi^{-1} : \Gamma - v \rightarrow \Gamma - u,$$

and

$$\psi : \Gamma - u \rightarrow \Gamma - v,$$

it follows that $\text{dom}(\theta) = \text{im}(\theta) = \Gamma - v$. Now consider the image of A under θ . From the definition of θ ,

$$\begin{aligned} A\theta &= A(\varphi^{-1}\psi) \\ &= (A\varphi^{-1})\psi \\ &= (N_\Gamma(u))\psi \\ &= N_\Gamma(v). \end{aligned}$$

Therefore, θ is an automorphism of $\Gamma - v$ mapping A onto $N_\Gamma(v)$. This proves our claim. □

2.4 If Γ is a graph on k vertices, what is the largest possible size of a set of mutually pseudo-similar in Γ ?

Notation. If $f, g : \mathbb{N} \rightarrow \mathbb{R}$ are real sequences, then the notation $f(n) = \mathcal{O}(g(n))$ means there exists a real number $C > 0$ and $N \in \mathbb{N}$ such that $|f(n)| \leq C|g(n)|$ for all $n \geq N$.

The following lemma is an important observation in characterising graphs with sets of mutually pseudo-similar vertices, and follows directly from [Lemma 2.7](#).

Lemma 2.9. *Let Γ be a regular graph. Then Γ does not contain a pair of pseudo-similar vertices.*

Proof. Suppose that the degree of any vertex in Γ is k . If $u \in V\Gamma$ then the set of neighbours $N_\Gamma(u)$ of u in Γ all have degree $k - 1$ in the graph $\Gamma - u$. A similar observation is true for any other vertex $v \in V\Gamma$. By a degree argument, any partial automorphism $\varphi : \Gamma - u \rightarrow \Gamma - v$ must induce a partial automorphism

$$\psi : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Therefore, by [Lemma 2.7](#), the partial automorphism φ can be extended to an automorphism φ' of Γ such that $u\varphi' = v$. It follows that if u, v are removal-similar in Γ , then they are similar. □

Theorem 2.10. *A graph Γ cannot have all its vertices mutually pseudo-similar.*

Proof. Let $u, v \in V\Gamma$ be such that $\Gamma - u \cong \Gamma - v$. Then $\deg_{\Gamma}(u) = \deg_{\Gamma}(v)$. It follows that if every vertex in Γ is removal-similar to every other vertex, then Γ is regular. Therefore, by [Lemma 2.9](#), Γ does not contain any pseudo-similar vertices. \square

We can do a little better.

Theorem 2.11. *Let Γ be a graph. Then Γ does not contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 1$.*

Proof. Let Γ be a graph. Let $A \subseteq V\Gamma$ be a set of mutually pseudo-similar vertices in Γ , and suppose for a contradiction that $|A| = |V\Gamma| - 1$.

Since all the vertices in A are removal-similar by assumption, $\deg_{\Gamma}(u) = k$ for all $u \in A$ and a fixed k . If $w \in V\Gamma \setminus A$ is the remaining vertex, then $\deg_{\Gamma}(w) \neq k$ since then Γ would be regular, so cannot contain pseudo-similar vertices by [Lemma 2.9](#). Let $\deg_{\Gamma}(w) = r \neq k$.

We consider 3 cases, depending on the value of r .

- In the case $r = 0$, then w is an isolated vertex of Γ . It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma - w$. By [Theorem 2.10](#), the vertices in A cannot be mutually pseudo-similar in $\Gamma - w$, and therefore cannot be mutually pseudo-similar in Γ . This is a contradiction.
- In the case $r = |V\Gamma| - 1$, consider the complement $\tilde{\Gamma}$ of Γ . By Corollary 6.5 in [\[JJSS21\]](#), Γ and $\tilde{\Gamma}$ have the same partial automorphism monoid, hence the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in $\tilde{\Gamma}$. If $r = |V\Gamma| - 1$, then w is an isolated vertex in $\tilde{\Gamma}$, and so the vertices in A cannot be pseudo-similar in $\tilde{\Gamma}$. This is a contradiction.
- In the case $0 < r < |V\Gamma| - 1$, there must exist $u, v \in A$ such that $u \notin N_{\Gamma}(w)$ but $v \in N_{\Gamma}(w)$. We show by a degree argument that $\Gamma - u \not\cong \Gamma - v$. First consider $\Gamma - u$. Since $w \notin N_{\Gamma}(u)$, the degree of w in $\Gamma - u$ is r . Moreover, for any $w' \in N_{\Gamma}(u)$, the vertex w' has degree $k - 1$ in $\Gamma - u$. It follows that the degree sequence of $\Gamma - u$ is

$$(r, \underbrace{k - 1, \dots, k - 1}_k, \underbrace{k, \dots, k}_{|V\Gamma| - k - 2}).$$

Now consider $\Gamma - v$. Since $w \in N_{\Gamma}(v)$, the degree of w in $\Gamma - v$ is $r - 1$. Given any $w' \in N_{\Gamma}(v) \setminus \{w\}$, the degree of w' in $\Gamma - v$ is $k - 1$. Therefore $\Gamma - v$ has degree sequence

$$(r - 1, \underbrace{k - 1, \dots, k - 1}_{k - 1}, \underbrace{k, \dots, k}_{|V\Gamma| - k - 1}).$$

Consider in particular the number of vertices of degree k . In the case $r = k + 1$, the graph $\Gamma - u$ has $|V\Gamma| - k - 2$ vertices of degree k , and $\Gamma - v$ has $|V\Gamma| - k$ vertices of degree k . Therefore, in this case, $\Gamma - u \not\cong \Gamma - v$. Moreover, if $r \neq k + 1$, then $\Gamma - u$ has $|V\Gamma| - k - 2$ vertices of degree k , and $\Gamma - v$ has $|V\Gamma| - k - 1$ vertices of degree k , so in this case also $\Gamma - u \not\cong \Gamma - v$. This is a contradiction.

□

It is therefore sensible to ask if this method can be extended to show that a graph Γ cannot contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$.

Theorem 2.12. *Let Γ be a graph. Then Γ does not contain a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$.*

The proof of [Theorem 2.12](#) is by contradiction. The argument is not simple, and is therefore structured into four main claims about Γ from which the desired contradiction is obtained. These are Propositions (a), (b), (c) and (d).

Suppose that Γ contains a set $A \subseteq V\Gamma$ of mutually pseudo-similar vertices, and that $|A| = |V\Gamma| - 2$. Assume $|A| \geq 2$. Let $w, w' \in V\Gamma \setminus A$ be the two remaining vertices. Then, there exists an integer k such that $\deg_{\Gamma}(u) = k$ for all $u \in A$. Notice that $k \geq 1$, otherwise the vertices in A would all be isolated vertices and hence would be similar. Let $\deg_{\Gamma}(w) = r$ and $\deg_{\Gamma}(w') = s$.

Our first claim about Γ is the following.

Proposition (a). For each $u \in A$, either $u \in N_{\Gamma}(w)$ or $N_{\Gamma}(w')$, but not both.

Proof. We prove this by showing that there cannot exist $u \in A$ such that both $u \notin N_{\Gamma}(w)$ and $u \notin N_{\Gamma}(w')$. The case where both $u \in N_{\Gamma}(w)$ and $u \in N_{\Gamma}(w')$ follows from the symmetric argument, or by considering the complement $\bar{\Gamma}$ of Γ .

Suppose that there exists $u \in A$ is such that $u \notin N_{\Gamma}(w)$ and $N_{\Gamma}(w')$. If $v \notin N_{\Gamma}(w)$ and $N_{\Gamma}(w')$ for all $v \in A$, then Γ is not connected, and A is a connected component of Γ . It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma[A]$. Therefore, by [Theorem 2.10](#), the vertices in A are not mutually pseudo-similar in Γ .

It follows that there exists $v \in A$ such that at least one of $v \in N_{\Gamma}(w)$ or $v \in N_{\Gamma}(w')$ is true. Suppose, without loss of generality, that $v \in N_{\Gamma}(w)$. We show, by a degree argument, that $v \notin N_{\Gamma}(w')$. Suppose that $v \in N_{\Gamma}(w')$. Since $u \notin N_{\Gamma}(w)$, the degree of w in $\Gamma - u$ is r . Similarly, the degree of w' in $\Gamma - u$ is s . Since $w, w' \notin N_{\Gamma}(u)$, any $u' \in N_{\Gamma}(u)$ has degree $k - 1$ in the graph $\Gamma - u$. Finally, any $u' \notin \{w, w'\} \cup N_{\Gamma}(u)$ has degree k in the graph $\Gamma - u$. Therefore, the degree sequence of $\Gamma - u$ is

$$(r, s, \underbrace{k - 1, \dots, k - 1}_k, \underbrace{k, \dots, k}_{|V\Gamma| - k - 3}).$$

By a similar argument, taking into account the fact that $w, w' \in N_{\Gamma}(v)$, the degree sequence of $\Gamma - v$ is

$$(r - 1, s - 1, \underbrace{k - 1, \dots, k - 1}_{k - 2}, \underbrace{k, \dots, k}_{|V\Gamma| - k - 1}).$$

Since $\Gamma - u \cong \Gamma - v$, these degree sequences must be equal. Hence, we must have $r = s = k$, so Γ is regular. This contradicts [Lemma 2.9](#). Therefore, it follows that $v \notin N_{\Gamma}(w')$.

Now, by a similar degree argument, we claim that $r = k$. By the previous argument, the degree sequence of $\Gamma - u$ is

$$(r, s, \underbrace{k-1, \dots, k-1}_k, \underbrace{k, \dots, k}_{|V\Gamma|-k-3}).$$

Moreover, taking into account the fact that $v \in N_\Gamma(w)$ and $v \notin N_\Gamma(w')$, it follows that the degree sequence of $\Gamma - v$ is

$$(r-1, s, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Since $\Gamma - u \cong \Gamma - v$, these degree sequences must be equal. This is only possible if $r = k$, proving our claim.

Now suppose that there exists $v' \in A$ such that $v' \notin N_\Gamma(w)$ and $v' \in N_\Gamma(w')$. By the symmetric argument, $s = k$, which implies that Γ is regular. This contradicts [Lemma 2.9](#). Moreover, since there cannot exist $v' \in A$ such that $v' \in N_\Gamma(w)$ and $v' \in N_\Gamma(w')$, we must conclude that

$$N_\Gamma(w') \setminus \{w\} = \emptyset.$$

There are now two cases to consider.

- In the case $N_\Gamma(w') = \emptyset$, the vertex w' is an isolated vertex in Γ . It follows that the vertices in A are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph $\Gamma - w'$. By [Theorem 2.11](#), the vertices in A are not mutually pseudo-similar in $\Gamma - w'$, therefore they are not mutually pseudo-similar in Γ . This is a contradiction.
- In the case $N_\Gamma(w') = \{w\}$, the graph Γ is of the form in [Figure 4](#). There are three sub-cases to be considered, depending on the value of k .
 1. In the case $k = 1$, each $u \in A$ is adjacent to only w . Therefore, Γ is a star graph with center w . In this case, all the vertices in A are similar, which is a contradiction.
 2. In the case $k = 2$, each vertex $u \in A$ is adjacent to w and exactly one $v \in A$. Since the same is true of the vertex v , it follows that there is an automorphism of Γ exchanging u and v . Therefore, $u \sim_S v$, which is a contradiction. This is illustrated in [Figure 5](#).
 3. In the case $k > 2$, suppose that $u, v \in A$ and

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. In the graph $\Gamma - u$, the vertex w' has degree 1, w has degree $|A| + 1 > 1$, each $u' \in N_\Gamma(u)$ has degree $k - 1 > 1$, and the remaining vertices have degree $k > 2$. A similar observation is true for the graph $\Gamma - v$. It follows that φ fixes w' , and since

$$N_{\Gamma-u}(w') = N_{\Gamma-v}(w') = \{w\},$$

φ fixes w . Consider the image of $N_\Gamma(u)$ under φ . By a degree argument, φ induces the partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Since φ fixes w , it then follows that φ induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Therefore, φ can be extended to an automorphism of Γ mapping u to v . This is a contradiction.

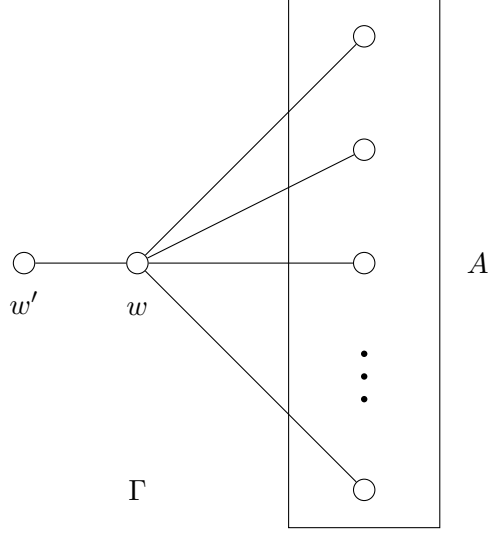


Figure 4: The graph Γ in the case $N_\Gamma(w') = \{w\}$.

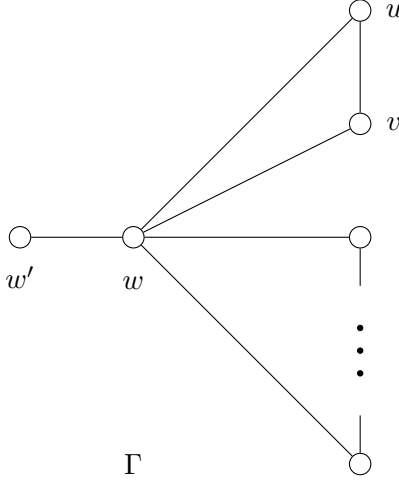


Figure 5: The graph Γ in the case $N_\Gamma(w') = \{w\}$ and $k = 2$.

Therefore, we have demonstrated that, in all cases, $N_\Gamma(w') \setminus \{w\} = \emptyset$ leads to a contradiction. The only conclusion left is that our initial assumption that there exists $u \in A$ such that $u \notin N_\Gamma(w)$ and $u \notin N_\Gamma(w')$ is false. Then, the fact that there cannot exist $u \in A$ such that $u \in N_\Gamma(w)$ and $u \in N_\Gamma(w')$ can then be deduced from considering the complement $\tilde{\Gamma}$ of Γ , since Γ and $\tilde{\Gamma}$ have the same partial automorphism monoid (Corollary 6.5 in [JJSS21]).

Therefore, for every $u \in A$, either $u \in N_\Gamma(w)$ or $u \in N_\Gamma(w')$, but not both. This proves Proposition (a).

□

Proposition (b). $r = s = k + 1$.

Proof. Let $u, v \in A$ be such that $u \in N_\Gamma(w)$ and $v \in N_\Gamma(w')$. Since, by Proposition (a), $u \notin N_\Gamma(w')$

and $v \notin N_\Gamma(w)$, it follows that the degree sequence of $\Gamma - u$ is

$$(r-1, s, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}),$$

and the degree sequence in $\Gamma - v$ is

$$(r, s-1, \underbrace{k-1, \dots, k-1}_{k-1}, \underbrace{k, \dots, k}_{|V\Gamma|-k-2}).$$

Since $\Gamma - u \cong \Gamma - v$, it follows that these degree sequences must be equal. Therefore, we must have $r = s$.

We now move on to showing that $r = k + 1$. The proof is by a degree argument, however the special case where $r = k - 1$ must be treated separately. Assume, for a contradiction, that $r \neq k + 1$.

Let $u, v \in A$ be such that $u, v \in N_\Gamma(w)$. Let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism.

In the graph $\Gamma - u$, each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k - 1$. Since $w \in N_\Gamma(u)$, the vertex w has degree $r - 1$ in the graph $\Gamma - u$, and since $w' \notin N_\Gamma(u)$, the vertex w' has degree r in the graph $\Gamma - u$. The remaining vertices all have degree k in $\Gamma - u$. A similar observation is true for the graph $\Gamma - v$.

Suppose first that $r \neq k - 1$. Since $r \neq k + 1$, the vertex w is the unique vertex of degree $r - 1$ in the graph $\Gamma - u$ and in the graph $\Gamma - v$. It follows that $w\varphi = w$. Moreover, since $r \neq k - 1$, the vertices in $N_\Gamma(u) \setminus \{w\}$ are only vertices in the graph $\Gamma - u$ of degree $k - 1$. A similar observation is true of $N_\Gamma(v) \setminus \{w\}$ in the graph $\Gamma - v$. Therefore, φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Thus, combining these two facts, φ induces a partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and therefore can be extended to an automorphism of Γ mapping u to v . This is a contradiction.

We now consider the special case when $r = k - 1$. First, in both $\Gamma - u$ and $\Gamma - v$, the vertex w is the unique vertex of degree $k - 2$, and so $w\varphi = w$. It follows that φ maps the set $N_{\Gamma-u}(w)$ onto the set $N_{\Gamma-v}(w)$. That is, φ induces a partial automorphism

$$\theta : N_\Gamma(w) \setminus \{u\} \rightarrow N_\Gamma(w) \setminus \{v\}.$$

Moreover, each $u' \in N_\Gamma(w') \setminus \{w\}$ has degree k in the graph $\Gamma - u$. Since $N_\Gamma(w') \setminus \{w\}$ is disjoint to both the sets $N_\Gamma(w) \setminus \{u\}$ and $N_\Gamma(w) \setminus \{v\}$, and moreover since $w\varphi = w$ and $\deg_{\Gamma-u}(w') = \deg_{\Gamma-v}(w') = k - 1$, it follows that $N_\Gamma(w') \setminus \{w\}$ is fixed setwise by φ .

Now consider the image of w' under φ . Since $r = k - 1$, the set of vertices of degree $k - 1$ in the graph $\Gamma - u$ is

$$\{w'\} \cup (N_\Gamma(u) \setminus \{w\}).$$

Similarly, the set of vertices of degree $k - 1$ in the graph $\Gamma - v$ is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Therefore, by a degree argument, φ induces a partial automorphism

$$\psi : \{w'\} \cup (N_\Gamma(u) \setminus \{w\}) \rightarrow \{w'\} \cup (N_\Gamma(v) \setminus \{w\}).$$

Using the fact that $w\varphi = w$, it follows that φ induces a partial automorphism

$$\psi' : \{w'\} \cup N_\Gamma(u) \rightarrow \{w'\} \cup N_\Gamma(v).$$

Let $u' \in \{w'\} \cup N_\Gamma(u)$ be such that $u'\varphi = w'$. If $u' = w'$ then we are done. Otherwise, $w'\varphi \in N_\Gamma(v)$, and so the image of $N_\Gamma(u)$ under φ is

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w'\varphi\}).$$

Consider the neighbours of w' in the graph $\Gamma - u$.

If w and w' are adjacent, then

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

It follows, since w is fixed by φ and $N_\Gamma(w') \setminus \{w\}$ is fixed setwise by φ , that

$$N_{\Gamma-v}(w'\varphi) = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Moreover,

$$N_{\Gamma-u}(w') = \{w\} \cup (N_\Gamma(w') \setminus \{w\}).$$

Similarly, in the case that w and w' are not adjacent,

$$N_{\Gamma-u}(w') = N_\Gamma(w') = N_\Gamma(w') \setminus \{w\}$$

which is fixed setwise by φ , and so

$$N_{\Gamma-v}(w'\varphi) = N_\Gamma(w') \setminus \{w\} = N_{\Gamma-u}(w').$$

Therefore, $\Gamma - v$ has an automorphism which exchanges w' and $w'\varphi$, and fixes every other vertex pointwise. This is an automorphism of Γ mapping

$$\{w'\} \cup (N_\Gamma(v) \setminus \{w'\varphi\}) = (N_\Gamma(u))\varphi$$

to $N_\Gamma(v)$, and hence, by [Theorem 2.8](#), u and v are similar in Γ . This is a contradiction.

Therefore, we must conclude that $r = s = k + 1$. This proves Proposition (b). □

Proposition (c). Each $u \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$.

Proof. Let $u \in A$. Since $N_\Gamma(w) \cap N_\Gamma(w') = \emptyset$ and $(N_\Gamma(w) \cup N_\Gamma(w')) \setminus \{w, w'\} = A$, we can partition the set $N_\Gamma(u) \setminus \{w\}$ as follows

$$N_\Gamma(u) \setminus \{w\} = N_\Gamma^w(u) \cup N_\Gamma^{w'}(u),$$

where $N_\Gamma^w(u)$ is the set of neighbours of u in A which are adjacent to w , and $N_\Gamma^{w'}(u)$ is the set of neighbours of u in A which are adjacent to w' .

Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. In the graph $\Gamma - u$, each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k - 1$, whereas every vertex not in this set has degree at least k . A similar observation is true in the graph $\Gamma - v$. Therefore, by a degree argument, φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Consider $w'\varphi$. Since $w' \notin N_\Gamma(u)$, the vertex w' has degree $k + 1$ in the graph $\Gamma - u$. Every other vertex in $\Gamma - u$ has degree k or $k - 1$. A similar observation is true of w' in the graph $\Gamma - v$. Therefore, by a degree argument, $w'\varphi = w'$.

Since φ fixes w' , it follows that φ maps the set $N_{\Gamma-u}(w')$ onto the set $N_{\Gamma-v}(w')$, and hence the set

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

is fixed setwise. Therefore, φ must induce partial automorphisms

$$\begin{aligned} \theta : N_\Gamma^w(u) &\rightarrow N_\Gamma^w(v) \\ \theta' : N_\Gamma^{w'}(u) &\rightarrow N_\Gamma^{w'}(v). \end{aligned}$$

Hence, $N_\Gamma^w(u) \cong N_\Gamma^w(v)$ and $N_\Gamma^{w'}(u) \cong N_\Gamma^{w'}(v)$, and so

$$\begin{aligned} |N_\Gamma^w(u)| &= |N_\Gamma^w(v)| \\ |N_\Gamma^{w'}(u)| &= |N_\Gamma^{w'}(v)|. \end{aligned}$$

Since u and v were arbitrary vertices in $N_\Gamma(w) \setminus \{w'\}$, it follows that there exist integers p and q such that, for all $u \in N_\Gamma(w) \setminus \{w'\}$,

$$|N_\Gamma^w(u)| = p \text{ and } |N_\Gamma^{w'}(u)| = q.$$

By the symmetric argument, with $u, v \in N_\Gamma(w') \setminus \{w\}$, it also follows that, for all $u \in N_\Gamma(w') \setminus \{w\}$,

$$|N_\Gamma^{w'}(u)| = p \text{ and } |N_\Gamma^w(u)| = q.$$

Now suppose that $q = 0$. We distinguish between the case when w, w' are adjacent and when w, w' are not adjacent.

- In the case where w, w' are not adjacent, then Γ is not connected. The two connected components of Γ are $N_\Gamma(w) \cup \{w\}$ and $N_\Gamma(w') \cup \{w'\}$. This is illustrated in [Figure 6](#). Consider, in particular, the component $N_\Gamma(w) \cup \{w\}$. The vertices in $N_\Gamma(w)$ are mutually pseudo-similar in Γ if and only if they are mutually pseudo-similar in the graph

$$\Gamma[N_\Gamma(w) \cup \{w\}].$$

By [Theorem 2.11](#), they cannot be mutually pseudo-similar in the graph $\Gamma[N_\Gamma(w) \cup \{w\}]$, hence they are not mutually pseudo-similar in Γ . This is a contradiction.

- In the case where w, w' are adjacent, first notice that we must have $|A| \geq 4$, so that $|N_\Gamma(w) \setminus \{w'\}| = |N_\Gamma(w') \setminus \{w\}| \geq 2$. This is illustrated in **Figure 7**. Otherwise, Γ is a path graph with an automorphism exchanging $N_\Gamma(w) \setminus \{w'\}$ and $N_\Gamma(w') \setminus \{w\}$, making the vertices in A similar, which is a contradiction.

Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and let

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

be a partial automorphism. In the graph $\Gamma - u$, the vertex w' has degree $k+1$; each $u' \in N_\Gamma(w')$ has degree k ; the vertex w has degree k ; each $u' \in N_\Gamma(w) \setminus (N_\Gamma(u) \cup \{w'\})$ has degree k ; and each $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k-1$. A similar observation is true for the graph $\Gamma - v$. It follows that φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Moreover, since w' is the only vertex of degree $k+1$ in both $\Gamma - u$ and $\Gamma - v$, it follows that $w'\varphi = w'$. Hence, φ fixes

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

setwise. Consider $w\varphi$. Since $w \in N_\Gamma(w')$, it follows that

$$w\varphi \in N_\Gamma(w').$$

Since $N_\Gamma(w) \geq 2$, the vertex w has a neighbour $u' \in N_\Gamma(w) \setminus \{w'\}$ which has degree $k-1$ in the graph $\Gamma - u$. A similar observation is true for the graph $\Gamma - v$. In contrast, since $q = 0$, each $u' \in N_\Gamma(w') \setminus \{w\}$ has no neighbour in the set $N_\Gamma(w) \setminus \{w'\}$, therefore each neighbour of u' has degree k or $k+1$ in the graph $\Gamma - u$. Similarly in the graph $\Gamma - v$. It follows that, if φ is a partial automorphism, $w\varphi = w$, and therefore φ induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v).$$

Hence, φ can be extended to an automorphism of Γ mapping u to v , making u and v similar. This is a contradiction.

Therefore, $q \geq 1$, so every $u \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$, as desired. This proves Proposition (c).

Proposition (d). Each $u \in N_\Gamma(w') \setminus \{w\}$ has at least one neighbour in $N_\Gamma(w) \setminus \{w'\}$.

Continuing from the proof of Proposition (c), suppose instead that $p = 0$. Let $u, v \in N_\Gamma(w) \setminus \{w'\}$, and suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. Once again, by a degree argument φ fixes w' , and hence fixes $N_\Gamma(w')$ setwise. Moreover, by a degree argument, φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

- In the case w, w' are not adjacent, since $N_\Gamma(w')$ is fixed setwise, it follows that $N_\Gamma(w) \setminus \{u\}$ is mapped onto $N_\Gamma(w) \setminus \{v\}$, and hence w is fixed. It follows that φ induces a partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and so φ can be extended to an automorphism of Γ mapping u to v . This implies $u \sim_S v$ in Γ , which is a contradiction.

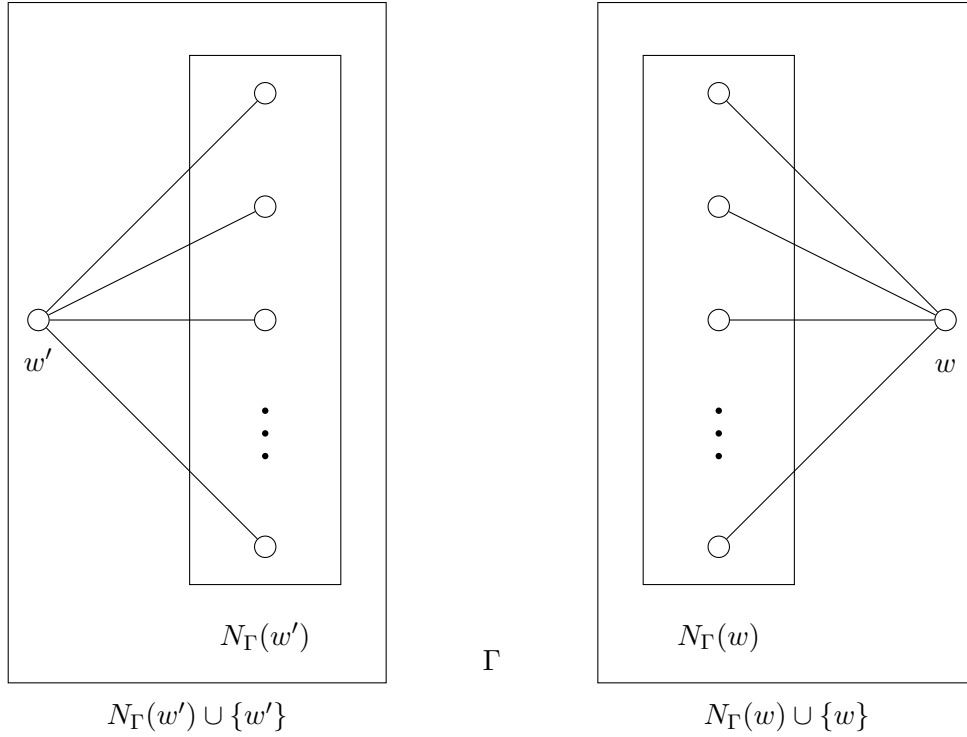


Figure 6: The graph Γ in the case that w, w' are not adjacent.

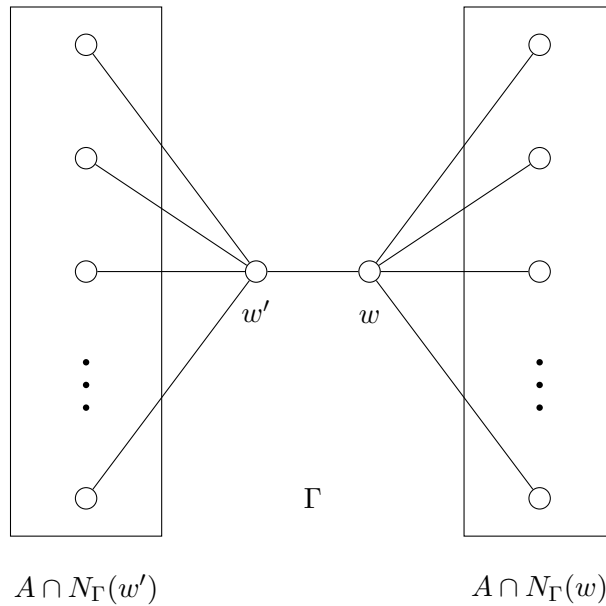


Figure 7: The graph Γ in the case that w, w' are adjacent.

- In the case w, w' are adjacent, since $N_{\Gamma}(w')$ is fixed setwise, it follows that $w\varphi \in N_{\Gamma}(w')$. Moreover, by a degree argument, $w\varphi \in \{w\} \cup (N_{\Gamma}(w') \setminus N_{\Gamma}(v))$ in the graph $\Gamma - v$. In the graph $\Gamma - v$, the vertex w is adjacent to w and all of the $k - 1$ vertices in the set $N_{\Gamma}(w) \setminus \{v\}$. Moreover, in the graph $\Gamma - v$ every $v' \in N_{\Gamma}(w') \setminus N_{\Gamma}(v)$ is adjacent to w' and all of the $k - 1$

vertices in the set $N_\Gamma(w) \setminus \{v, w'\}$. Hence,

$$N_{\Gamma-v}(w) = N_{\Gamma-v}(w\varphi),$$

and therefore $\Gamma - v$ has an automorphism which exchanges φ and $w\varphi$, and fixes all other vertices pointwise. This is an automorphism of $\Gamma - v$ mapping the set $(N_\Gamma(u))\varphi$ onto $N_\Gamma(v)$, and so by [Theorem 2.8](#), $u \sim_S v$ in Γ . This is a contradiction.

It follows that $p \geq 1$, and so each $u \in N_\Gamma(w') \setminus \{w\}$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. This proves Proposition (d). \square

Proof of Theorem 2.9. For the remainder of the argument, let $u, v \in A$ be such that $u, v \in N_\Gamma(w)$. Suppose that

$$\varphi : \Gamma - u \rightarrow \Gamma - v$$

is a partial automorphism. By Claim 2, the vertex w' has degree $k + 1$ in the graph $\Gamma - u$, and also has degree $k + 1$ in the graph $\Gamma - v$. Every vertex not equal to w' has degree at most k in $\Gamma - u$, and similarly in $\Gamma - v$. Thus, $w'\varphi = w'$. It follows that φ maps the set $N_{\Gamma-u}(w')$ onto the set $N_{\Gamma-v}(w')$, and therefore

$$N_\Gamma(w') = N_{\Gamma-u}(w') = N_{\Gamma-v}(w')$$

is fixed setwise.

Moreover, every $u' \in N_\Gamma(u) \setminus \{w\}$ has degree $k - 1$ in $\Gamma - u$. Similarly, every $v' \in N_\Gamma(v)$ has degree $k - 1$ in $\Gamma - v$. The remaining vertices all have degree at least k . It follows, by a degree argument, that φ induces a partial automorphism

$$\psi : N_\Gamma(u) \setminus \{w\} \rightarrow N_\Gamma(v) \setminus \{w\}.$$

Now consider $w\varphi$. By Proposition (b), w has degree $k + 1$ in Γ , and so has degree k in $\Gamma - u$. Similarly, w has degree k in $\Gamma - v$. Therefore, by a degree argument, either

$$w\varphi \in N_\Gamma(w'),$$

or

$$w\varphi \in \{w\} \cup N_\Gamma(w) \setminus (\{w'\} \cup N_\Gamma(v)).$$

We again distinguish between the cases when w, w' are adjacent and w, w' are not adjacent.

- In the case w, w' are not adjacent, then $w\varphi \in \{w\} \cup N_\Gamma(w) \setminus N_\Gamma(v)$. Suppose that $w\varphi \neq w$. By Claim 3, every $u' \in N_\Gamma(w) \setminus \{w'\}$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. It follows that $w\varphi$ has at least one neighbour in $N_\Gamma(w') \setminus \{w\}$. However, w has no neighbour in $N_\Gamma(w') \setminus \{w\}$, and since $N_\Gamma(w')$ is fixed setwise by φ , this is a contradiction.
- In the case w, w' are adjacent, then $w\varphi \in N_\Gamma(w)$. Suppose for a contradiction that $w\varphi \neq w$. In $\Gamma - u$, the vertex w has no neighbours in the set $N_\Gamma(w') \setminus \{w\}$. However, by Claim 3, if $w\varphi \in N_\Gamma(w') \setminus \{w\}$, then $w\varphi$ has at least one neighbour in the set $N_\Gamma(w') \setminus \{w\}$. Denote this neighbour u' . Since φ fixes $N_\Gamma(w')$ setwise, it follows that $u'\varphi^{-1} \in N_\Gamma(w')$. Moreover, since $w\varphi$ and u' are adjacent, it follows that $u' \neq w\varphi$, and hence $u'\varphi^{-1} \neq w$. Therefore, $u'\varphi^{-1} \in N_\Gamma(w') \setminus \{w\}$, so w has a neighbour in the set $N_\Gamma(w') \setminus \{w\}$. This contradicts Proposition (a).

Therefore, $w\varphi = w$. It follows that φ induces the partial automorphism

$$\psi' : N_\Gamma(u) \rightarrow N_\Gamma(v),$$

and hence by [Lemma 2.7](#), $u \sim_S v$ in Γ . This contradicts their pseudo-similarity. It follows that there cannot exist a graph Γ containing a set of mutually pseudo-similar vertices of size $|V\Gamma| - 2$. This proves [Theorem 2.12](#). □

The method of reduction used to prove [Theorem 2.12](#) is clearly limited. As we decrease $|A|$ relative to $|V\Gamma|$, the amount of case analysis required rapidly becomes untenable. In fact, the method will eventually fail entirely: there exists a graph Γ with a set of mutually pseudo-similar vertices of size $|V\Gamma| - 6$. For an example, see [Figure 9](#).

One possible route for further progress may be to consider generalisations of Proposition (a). If such a generalisation could be used to show that, for each $u \in A$, a sufficiently large fraction of the vertices in $N_\Gamma(u)$ are not pseudo-similar to u in Γ , then this could be used to prove a new lower bound for [Question 1.3](#).

2.5 Digraphs and constructing graphs with arbitrarily large sets of mutually pseudo-similar vertices

The case when Γ is digraph contrasts sharply with the case when Γ is a graph. For example, [Theorem 2.10](#) is false for digraphs: there exist digraphs with every vertex mutually pseudo-similar to every other vertex. An important example is the *transitive tournament*, T_k , defined on the vertex set

$$VT_k = \{1, 2, \dots, k\}$$

with $u \rightarrow v$ if and only if $u < v$. For a picture, see [Figure 8](#).

Proposition 2.13. *In T_k , every vertex is mutually pseudo-similar to every other vertex.*

Proof. First, note that T_k has no automorphism other than the identity, so no pair of distinct vertices $u, v \in VT_k$ is similar. By considering T_k as a linear order on the vertices $\{1, 2, \dots, k\}$, we can see that $T_k - u \cong T_{k-1}$ for every $u \in VT_k$. Therefore, for every pair $u, v \in T_k$, we have $T_k - u \cong T_k - v$. Thus, any pair $u, v \in T_k$ is pseudo-similar. □

Given a transitive tournament T_k , it is possible to replace the directed edges with undirected ‘gadgets’ while maintaining the mutual pseudo-similarity of the vertices. We describe on such construction, due to [\[KJSS81\]](#). Consider the undirected gadget Γ in [Figure 9](#). The vertices u and v are pseudo-similar in Γ , and, moreover, are cutvertices of Γ , that is, the graphs $\Gamma - u$ and $\Gamma - v$ are not connected. For each pair $u, v \in VT_k$, replace the directed edge $u \rightarrow v$ with the gadget Γ , including the extra vertices on each side. See [Figure 10](#) for an example. This produces a sequence of graphs, Γ_k , containing a set of k mutually pseudo-similar vertices and

$$|V\Gamma_k| = k + (|V\Gamma| - 2) \binom{k}{2} = k + 6 \binom{k}{2} = \mathcal{O}(k^2).$$

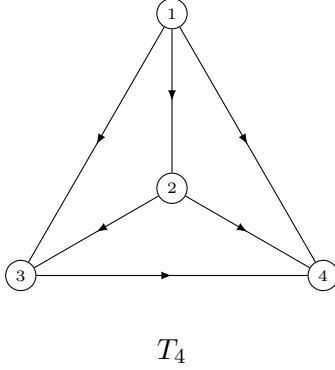


Figure 8: The transitive tournament T_4 .

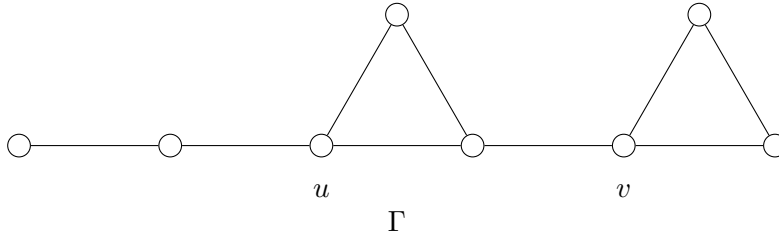


Figure 9: The gadget Γ , with pseudo-similar cutvertices u and v .

The problem with transforming T_k into an undirected graph in this manner is that, while every vertex in T_k is mutually pseudo-similar to every other vertex, and hence contains a set of k mutually pseudo-similar vertices, T_k contains $\binom{k}{2} = \mathcal{O}(k^2)$ edges. When replacing a directed edge with a gadget, at least one more vertex is added, so the resulting undirected graph has $\mathcal{O}(k^2)$ vertices.

A natural question is therefore to ask, does there exist a sequence of digraphs with every vertex mutually pseudo-similar to every other vertex and with fewer than $\mathcal{O}(k^2)$ directed edges?

Problem 2.14. *Let Γ_k be a digraph with $|V\Gamma_k| = k$ such that every vertex is mutually pseudo-similar to every other vertex. Does this then imply $\Gamma_k \cong T_k$?*

This problem is deceptively difficult. One subtlety occurs when considering extensions of partial automorphisms of the form

$$\varphi : \Gamma_k - u \rightarrow \Gamma_k - v$$

to automorphisms of Γ_k . Unlike in graphs, it is not sufficient for φ to be able to be extended to an automorphism of Γ_k that φ maps the (undirected) neighbours of u to the (undirected) neighbours of v . It also required that φ respects the partition of $N_\Gamma(u)$ and $N_\Gamma(v)$ into the in-neighbours and out-neighbours. This subtlety causes degree arguments of the kind used in [Lemma 2.9](#) to lose their power.

Consider, for instance, the following proposition about extending partial automorphisms of digraphs, which may be viewed as an attempt at an analogue of [Lemma 2.7](#) for digraphs.

Proposition 2.15. *Let Γ be a digraph, and $u, v \in V\Gamma$ be removal similar. Suppose that $\Gamma - u - v$ has no automorphism mapping an in-neighbour of u to an out-neighbour of u , and vice-versa. Let*

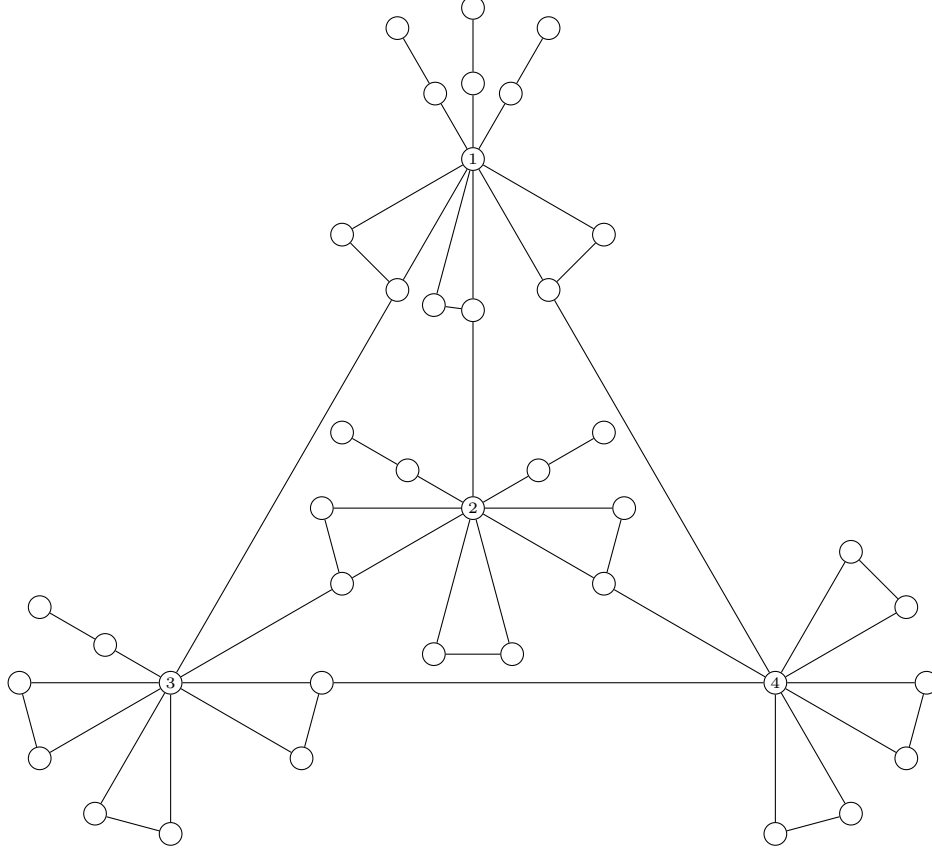


Figure 10: The transitive tournament T_4 , with directed edges replaced with appropriate undirected gadgets.

$\varphi : \Gamma - u \rightarrow \Gamma - v$ be a partial automorphism, and suppose that restricting φ induces two partial automorphisms

$$\begin{aligned}\psi &: N_{\Gamma}(u) \rightarrow N_{\Gamma}(v) \\ \theta &: N'_{\Gamma}(v) \rightarrow N'_{\Gamma}(u)\end{aligned}$$

such that $v\theta = u$. Then u and v are similar in Γ .

Proof. Let Γ , u and v be as given in the statement of the lemma, and let $\varphi : \Gamma - u \rightarrow \Gamma - v$ be a partial automorphism which induces two partial automorphisms

$$\begin{aligned}\psi &: N_{\Gamma}(u) \rightarrow N_{\Gamma}(v) \\ \theta &: N'_{\Gamma}(v) \rightarrow N'_{\Gamma}(u)\end{aligned}$$

Notice that, since $\varphi : \Gamma - u \rightarrow \Gamma - v$, and $v\varphi = u$, it follows that φ^2 is an automorphism of $\Gamma - u - v$. Suppose, for a contradiction, that there is a $w \rightarrow u$ such that $v \rightarrow w\varphi$. Then, $w\varphi^2 = (w\varphi)\theta$, so if $v \rightarrow w\varphi$ then $v\theta \rightarrow (w\varphi)\theta$ and so $u \rightarrow w\varphi^2$. Therefore, φ^2 is an automorphism of $\Gamma - u - v$ which maps an in-neighbour of u to an out-neighbour of u , which is a contradiction. Since, in addition to this, φ induces the map

$$\psi : N_{\Gamma}(u) \rightarrow N_{\Gamma}(v)$$

it follows that φ must map in-neighbours of u to in-neighbours of v , and by the symmetric argument, map out-neighbours of u onto out-neighbours of v . It follows that φ can be extended to an automorphism of Γ mapping u to v , so $u \sim_S v$ in Γ . \square

Proposition 2.5 is weak in the sense that the requirements placed on Γ and φ are very strong. If **Question 2.14** is to be approached, then these conditions must be weakened significantly.

Another approach is to weaken the condition that $|V\Gamma_k| = k$. If instead, we only require that Γ_k contains a set of k mutually pseudo-similar vertices with $|V\Gamma_k| = \mathcal{O}(k)$, then a wider class of digraphs may be considered.

Problem 2.16. *Construct a sequence of digraphs Γ_k such that*

1. Γ_k contains a set of k mutually pseudo-similar vertices;
2. $|V\Gamma_k| = \mathcal{O}(k)$; and
3. $|E\Gamma_k| = \mathcal{O}(k)$.

If such a sequence Γ_k were found, then it could possibly be transformed using undirected gadgets to produce a sequence of graphs having a set of k mutually pseudo-similar vertices and $\mathcal{O}(k)$ vertices.

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