Understanding Pseudo-similarity in Graphs: a Path to Proving the Reconstruction Conjecture

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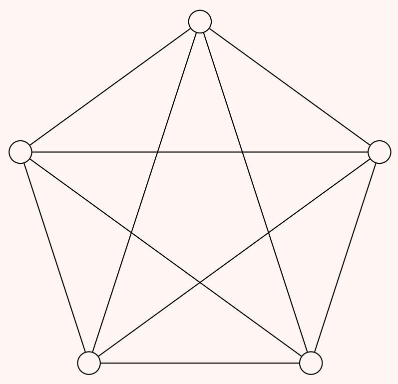
Graphs, Symmetries and Partial Symmetries

A *graph* is an ordered pair (V, E), where V is the *vertex* set, and $E \subseteq [V]^2$ is the *edge* set, a collection of 2-element subsets of V. Graphs are fundamental mathematical structures which are used to abstractly model the connections between objects or ideas.



K_5

Figure 1. The complete graph K_5, whose automorphism group is the symmetric group S_5.

Mathematicians have studied the *symmetries* of graphs for centuries. By a symmetry of a graph Γ, we mean an *automorphism* φ of Γ, which is a relabeling of the graph's vertices which preserves the edges and non-edges. Formally, it is a permutation φ of V which fixes E setwise. Studying symmetry is a vital tool for a mathematician when understanding graphs and their properties.

The idea of an automorphism of Γ can be generalised to a *partial* automorphism of Γ, which is an isomorphism between vertex-induced subgraphs of Γ. While the automorphisms of a graph Γ form a *group*, denoted $\mathrm{Aut}(\Gamma)$, the partial automorphisms form an *inverse monoid*, denoted $\mathrm{PAut}(\Gamma)$, which may be seen as a generalisation of the graph's automorphism group [2].

Pseudo-Similarity

Two vertices u and v in a graph Γ are called *similar* if there is an automorphism φ of Γ such that $u\varphi = v$. Intuitively, vertices are similar if they are indistinguishable when the vertex labels are removed.

Two vertices u and v are called *removal-similar* if the vertex-deleted subgraphs $\Gamma - u$ and $\Gamma - v$ are isomorphic. If u and v are similar, then they are removal-similar, since the automorphism φ can be restricted to give an isomorphism from $\Gamma - u$ to $\Gamma - v$.

However, **the converse is not true**. Two vertices u and v can be removal-similar in Γ, but not similar. In this case, they are called *pseudo-similar*.

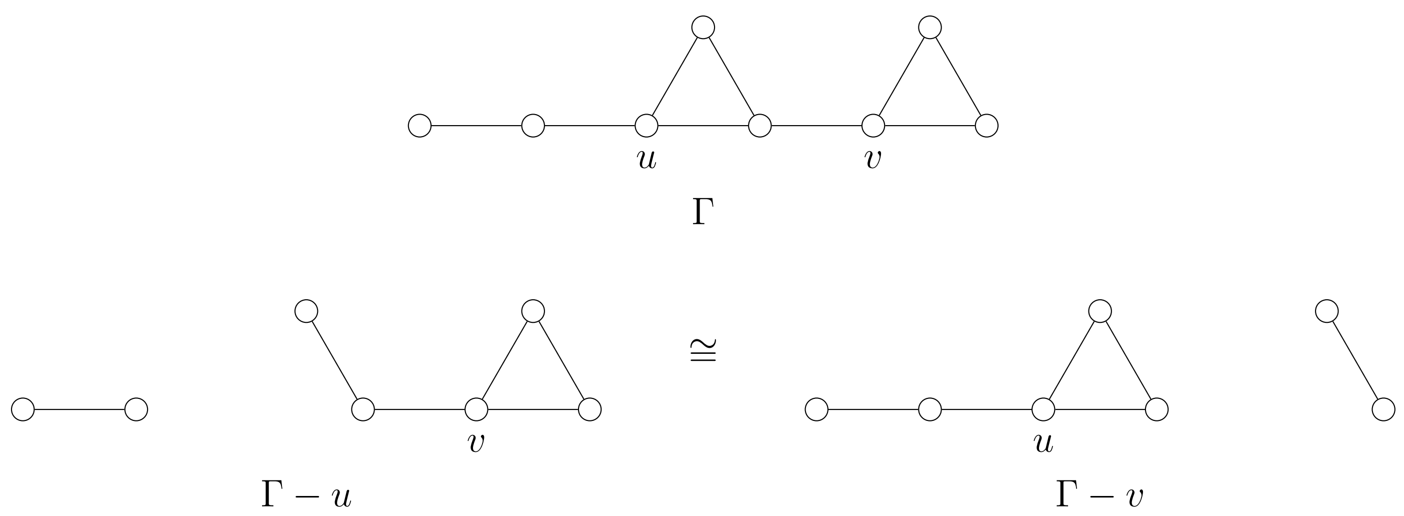


Figure 2. The graphs $\Gamma - u$ and $\Gamma - v$ are isomorphic, but no automorphism of Γ maps u to v, hence u and v are pseudo-similar in Γ.

Vertices $\{v_1, v_2, \ldots, v_k\} \subseteq V$ are called *mutually pseudo-similar* if they are pairwise pseudo-similar.

Question. For each $k \geq 2$, what is the size of the smallest graph containing a set of k mutually pseudo-similar vertices?

The Reconstruction Conjecture

Given a graph Γ, define the deck of Γ, denoted $\mathrm{Deck}(\Gamma)$, to be the multiset of isomorphism classes of the graphs $\Gamma - u$ for each $u \in V$. The Reconstruction Conjecture claims that every finite graph on at least three vertices is uniquely reconstructible from its deck; that is, given any two finite graphs Γ and Γ' on at least three vertices, $\mathrm{Deck}(\Gamma) \cong \mathrm{Deck}(\Gamma')$ if and only if $\Gamma \cong \Gamma'$.

The Reconstruction Conjecture is one of the most important long-standing open problems in graph theory. Importantly, a 'proof' of the conjecture emerged whose error was the false assumption that pseudo-similar vertices do not exist [1]. A compelling motivation to study pseudo-similar vertices is therefore to better understand, and resolve, the Reconstruction Conjecture.

A New Upper Bound

Previously, it was known that a graph Γ cannot contain a set of mutually pseudo-similar vertices of size $|V| - 1$. Using the ideas of partial automorphisms, I was able to extend this to show that a graph Γ cannot contain a set of mutually pseudo-similar vertices of size $|V| - 2$.

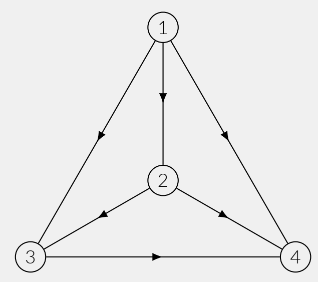
Proof Strategy

1. Suppose for a contradiction that Γ is a graph containing a set A of mutually pseudo-similar vertices of size $|V| - 2$. Let the remaining vertices be w and w'.
2. It can be shown, by a degree argument and considering partial automorphisms, that for every $u \in A$, the vertex u is adjacent to either w or w', but not both.
3. Since the vertices in A are mutually pseudo-similar, they must have some common degree k. It can be shown, by considering the partial automorphisms of Γ, that w and w' both then have degree $k + 1$.
4. These facts can then be used to show, by considering a further degree argument, that for any $u, v \in A$, any partial automorphism of the form $\varphi : \Gamma - u \to \Gamma - v$ must map the neighbours of u onto the neighbours of v, and therefore can be extended to an automorphism of Γ mapping u to v. This is a contradiction.

$\square$

An Important Construction

The *transitive tournament* T_k is a directed graph on the vertex set $V = \{1, 2, \ldots, k\}$, where the directed edge $(i, j) \in E$ if $i < j$. T_k is important as all of its vertices are mutually pseudo-similar.



T_4

Figure 3. The transitive tournament T_4.

By replacing the directed edges with undirected 'gadgets', we can form an undirected graph with a set of k mutually pseudo-similar vertices and $\mathcal{O}(k^2)$ vertices [3]. I investigated constructions involving fewer total vertices, beginning by considering digraphs with pseudo-similar vertices and fewer directed edges.

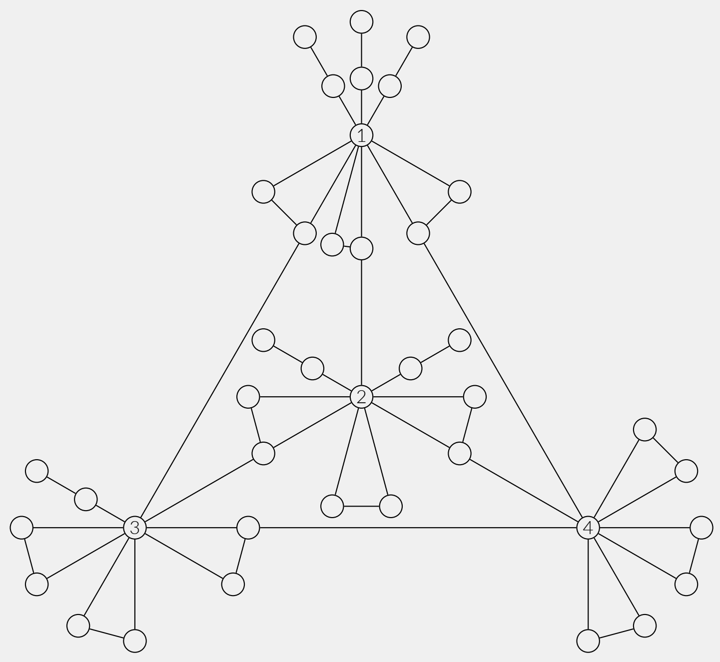


Figure 4. The transitive tournament T_4, with directed edges replaced with appropriate undirected gadgets. The vertices labelled 1, 2, 3, and 4 are all mutually pseudo-similar.

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