PSEUDOSPECTRA AND UTMOST RIGHT NORM

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Introduction

Stability of Linear Systems

- Model: **dx/dt = A x**
- Stable if all **Re(λ(A)) < 0**

Problem

- For **non-normal matrix A** (A*A' ≠ A'*A):
  - Eigenvalues can be misleading
    - Instability is not visible
  - Small errors lead to large asymptotic growth

Solution

- Introduce **pseudospectra** to show sensitivity
- Plot stability boundaries under perturbations
- Use **randomized algorithms** for:
  - Faster computation
  - Large-scale matrices
  - Stable implementations

A bit of theory

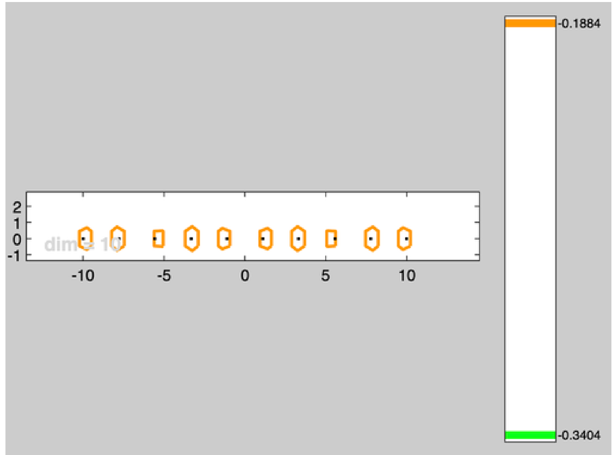
Pseudospectrum (σ_ε(A))

- Set of eigenvalues of **A+E** with ‖E‖ < ε
- Three equivalent definitions:
  - **Perturbed spectra**: eigenvalues of A+E
  - **Resolvent norm**: $\|(zI-A)^{-1}\| > 1/\varepsilon$
  - **Approximate eigenvector**: z ∈ σ_ε(A) if there exists a unit vector v such that ||(A - zI)v|| < ε.
- **Normal matrix A:** σ_ε(A) is ε-neighborhood of eigenvalues
- **Non-normal matrix A:**σ_ε(A) can diverge

Pseudospectral Abscissa

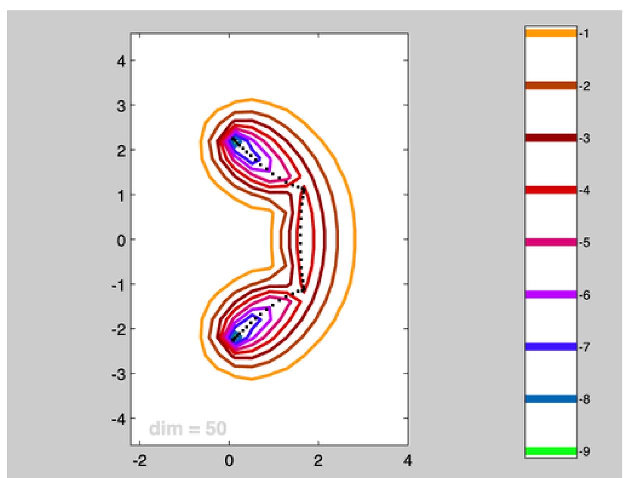
- **α_ε(A) = max Re(z), z ∈ σ_ε(A)**
- Worst-case growth rate under perturbations

Examples on MATLAB



100×100 diagonal normal matrix

- Eigenvalues: black dots on real axis
- **ε-pseudospectra** : circular contours around each eigenvalue
- For normal matrices: stability **insensitive** to small perturbations



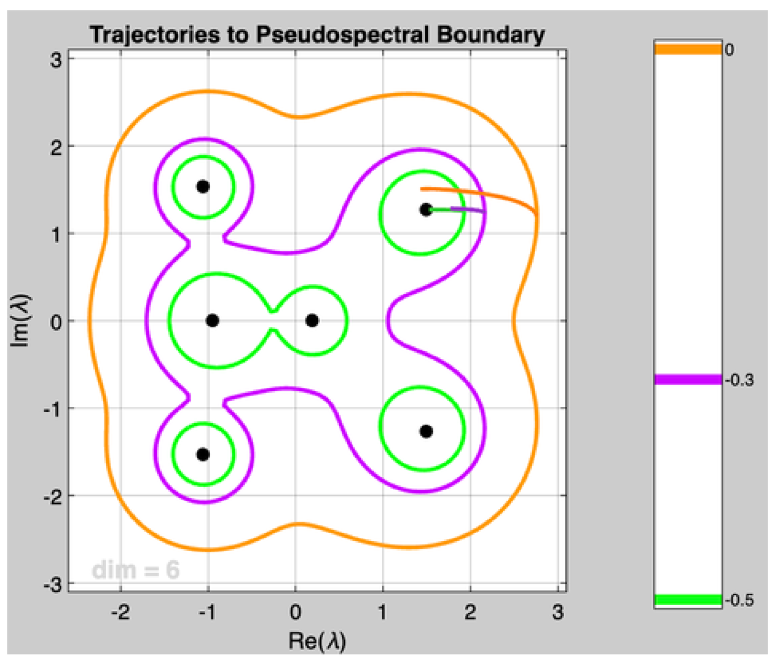
Grcar 50x50 Toeplitz non-normal matrix

- Stable eigenvalues
- **ε-pseudospectra** goes into right-half plane
- Even tiny perturbations lead to potential instability
- Non normal matrices are sensible to perturbations

Tracking the Pseudospectral Boundary

Algorithm: Rightmost Pseudospectral Point

- Start with random unit vectors **u, v**
- Perturb matrix: **B = A + ε u v***
- Find rightmost eigenvalue **λ** of B (with eigenvectors x, y)
- Project **x y*** onto tangent space to get **du/dt** and **dv/dt**
- Update u, v via Euler step;
- Repeat the process
- **Trajectory of λ(t):** moves monotonically right and converges to boundary point with largest **Re(λ)**
- Reveals **worst-case instability** under ε-perturbations



Low rank evolution methods

Problem

- Gugliemi- Lubich's approach:
  - Projection onto tangent space may deviate from **x y*** :
    - Tangent space is different from ambient space
    - Errors can accumulate
      - Drift from low rank structure for large scaled matrices

Solution: Two Randomized Methods

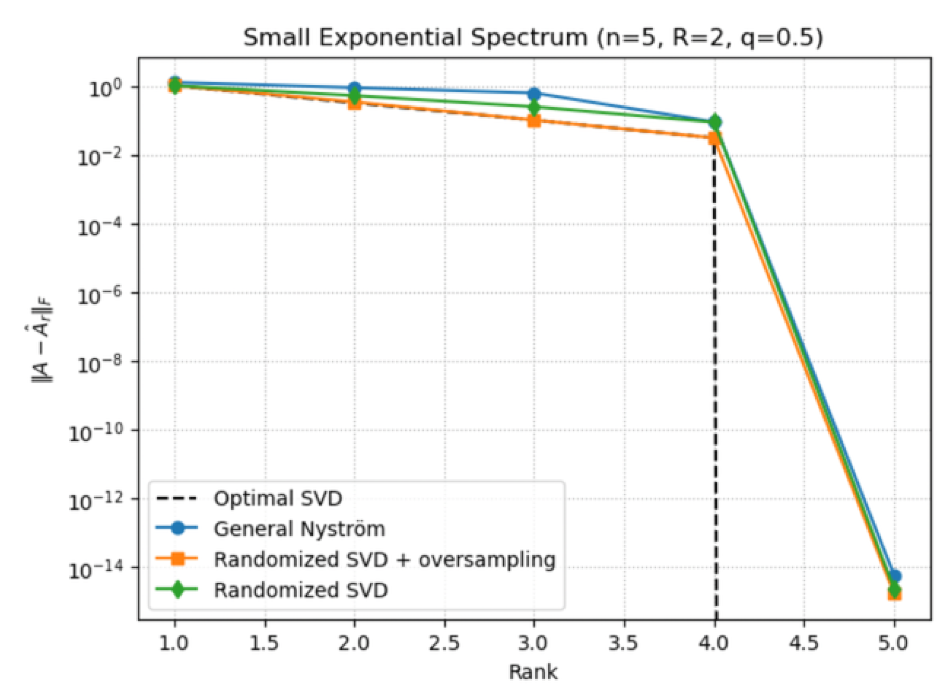
Project directly in the **ambient space**

- **Randomized SVD**
  - Captures dominant singular subspace efficiently
- **Generalized Nyström**
  - Builds low-rank approximation from rows and columns sketches

Frobenius norm error vs. rank for low-rank approximation methods.

Comparing Low-Rank Methods

- **Optimal SVD error** (black dashed line) is the theoretical lower bound
- **Randomized SVD** tracks optimal error
- **Generalized Nyström** also follows optimal error



Oversampling

We want randomized low-rank approximation with target rank k.
Problem: Random projections may miss important directions.
Solution : We sample k+p vectors instead of k

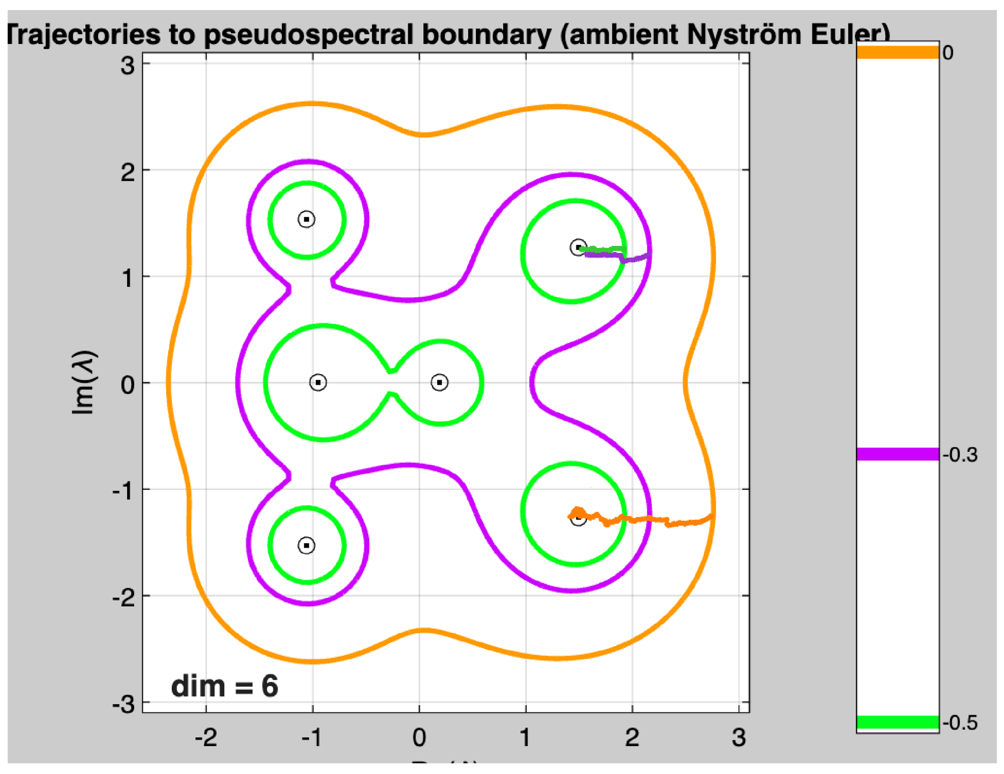
Randomized Euler

Enforces a fixed low rank r after each step, using matrix sketching with Nyström algorithm.

The algorithm:

1. Initialize $E_0 = u_0 v_0^*$ (rank-1).
2. Euler Step: Compute the update $Z = E_n + h\,F(E_n)$.
3. Nyström Projection: Project Z to rank-r: $E_{n+1} = Z\,\Omega\,(\Psi^T Z\,\Omega)^+\,\Psi^T Z$(where Ω, Ψ are random test matrices).
4. Iterate. Track the eigenvalue λ of $A + \varepsilon E_n$

Why it works: It avoids doing costly SVDs. It relies on fast matrix-vector products. Cost per step is O(n r²).



Trajectories to pseudospectral boundary with randomized Euler method

- Compress gradient update to rank 1 via Nyström and normalize
- Avoids manifold projections: More efficient than the first algorithm implemented
- Still shows pseudospectral boundaries and worst-case stability

Conclusion

- **Worst-case growth:** Rightmost ε-pseudospectral boundary
- **Guglielmi–Lubich:** Efficient vector updates but tangent projection may drift
- **Randomization:** Randomized SVD and generalized Nyström method
  - Cheap low-rank approximation
- **Randomized Euler:** Nyström compression keeps rank-one, avoids projections, more accurate by staying on ambient space

REFERENCES

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