

Formalizing the Max-Plus Spectral Theorem in Lean 4

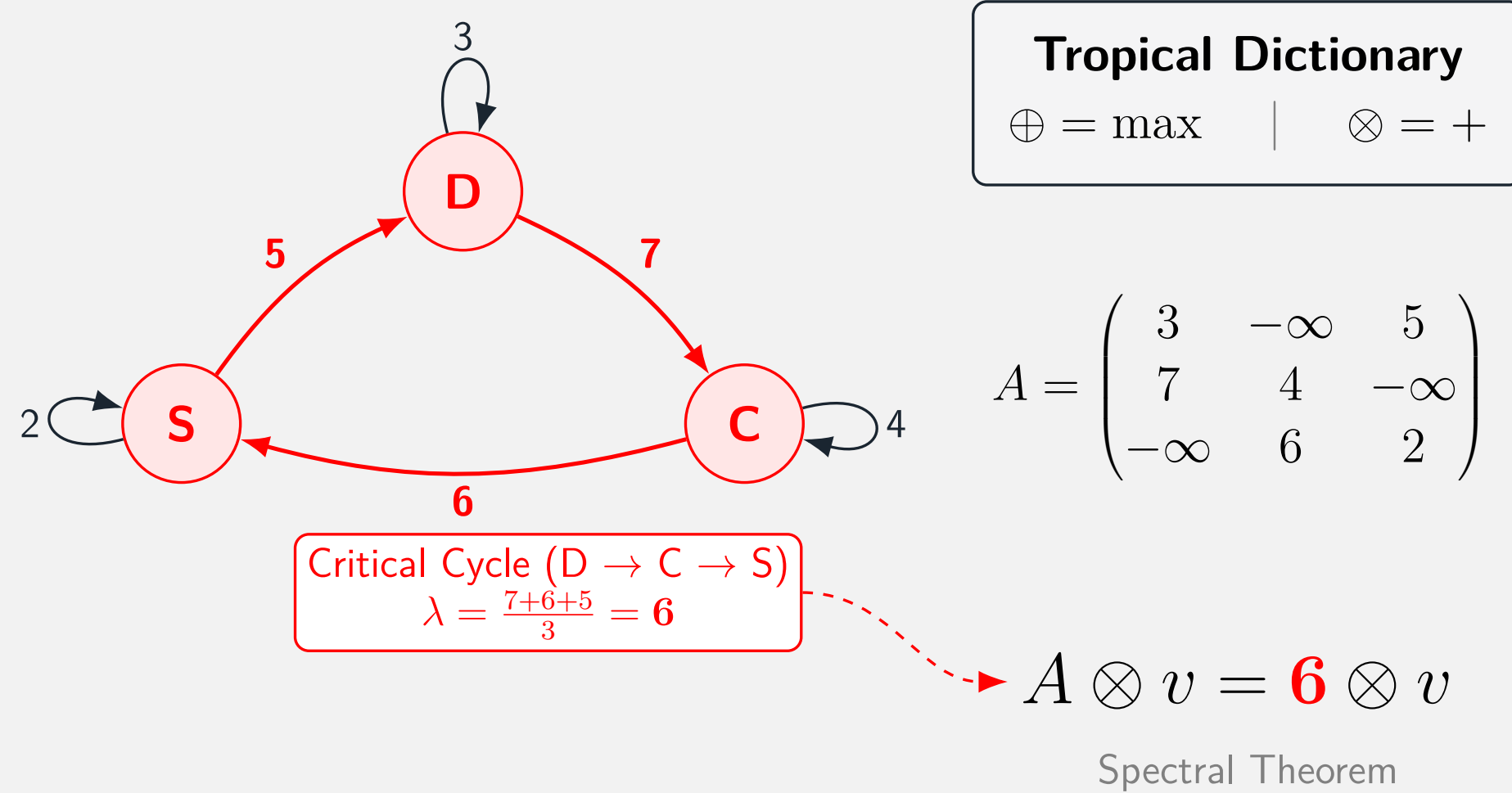
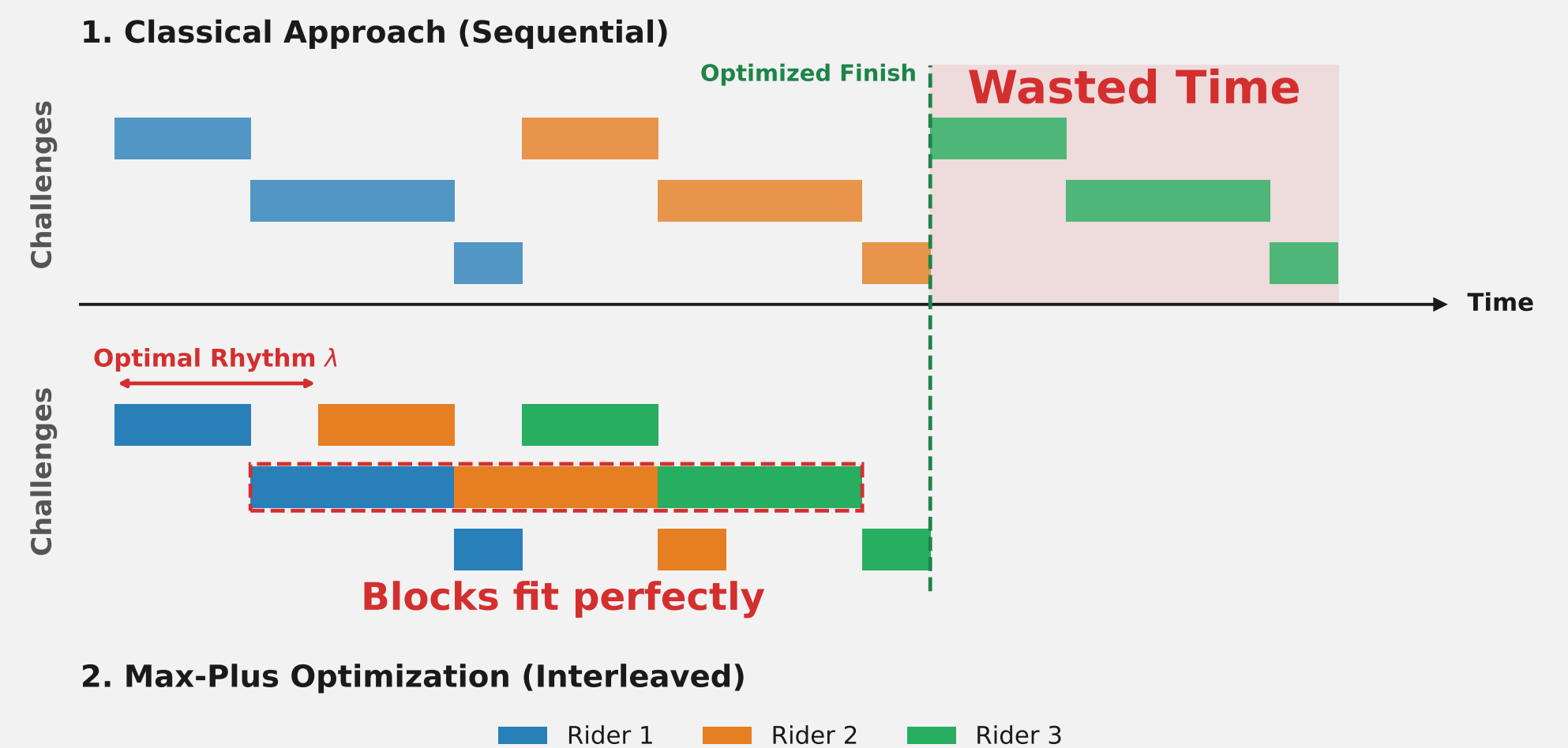
Why did an equestrian triathlon need tropical algebra to avoid chaos?

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1. The Equestrian Bottleneck

- **The incident (Equissima, Sept. 2025):** A fall on the track, and the schedule of the equestrian triathlon falls apart in an instant.
- **My solution:** An instant rescheduling algorithm powered by tropical (max-plus) algebra to detect bottlenecks.
- **The challenge:** How can we mathematically guarantee that an emergency-handling algorithm is 100% error-free?



2. Tropical Algebra

- **Max-Plus semiring:** addition becomes a maximum ($a \oplus b = \max(a, b)$) to represent waiting for constraints, and multiplication becomes addition ($a \otimes b = a + b$) to accumulate durations.
- **Linearizing Time:** Scheduling rules are reduced to a linear matrix recurrence: $x(k) = A \otimes x(k-1)$.
- **The Spectral Theorem:** Solving $A \otimes v = \lambda \otimes v$ reveals the system's heartbeat. The eigenvalue $\lambda = \max_{c \in \mathcal{C}} \frac{\sum_{e \in c} w(e)}{|c|}$ is the bottleneck, the Maximum Cycle Mean, allowing $O(1)$ predictions of future events: $x(k+p) = x(k) + p \cdot \lambda$.

3. Formalization in Lean 4

- **My contribution:**
 - **Building the foundations:** Formalized the missing algebraic structures in mathlib, specifically the tropical semiring (max-plus algebra) and its matrix operations.
 - **Bridging algebra and graphs:** Established the formal link between tropical matrices and weighted directed graphs by encoding concepts like walks, paths, and circuits into Lean.
 - **The main proof:** Translated the informal paper proof of the Max-Plus Spectral Theorem into a 100% machine-checked proof, proving the existence of eigenvalues and critical circuits.
 - **The result:** A fully verified, error-free mathematical foundation ready to guarantee the safety of my rescheduling algorithm.

```
def IsMaxPlusEigenvalue (mu : RMax)
  : Prop :=
  ∃ v : ι → RMax, v ≠ 0 ∧ A *v v = fun
    i ↦ mu * v i

theorem
  existsUnique_maxPlusEigenvalue
  (hA : A.IsMaxPlusIrreducible) :
  ∃! mu : RMax, A.IsMaxPlusEigenvalue
    mu := by [scan for code and
  proof]
```

4. Impact

- **For mathematics:** A new building block of tropical algebra is now formally certified and open-source.
- **On the field:** Verified theory powers real-world safety: battle-tested successfully during the 2026 competition, it is now recommended in the official guide.



Project & App:
jeremydigard.com

Lean 4 Source Code:
github.com/jeremydigard/spectral_max_plus

