Informative Threshold Events for Lotka-Volterra Parameter Estimation

A variance-based event-selection and gradient-matching framework

Ved Lakshminarayan

Acknowledgement: I would like to thank my supervisor, Professor Wenying Shou, for their mentorship and Dr Roman Ulises Zapien Campos for their invaluable guidance and support in shaping the project.

Motivation

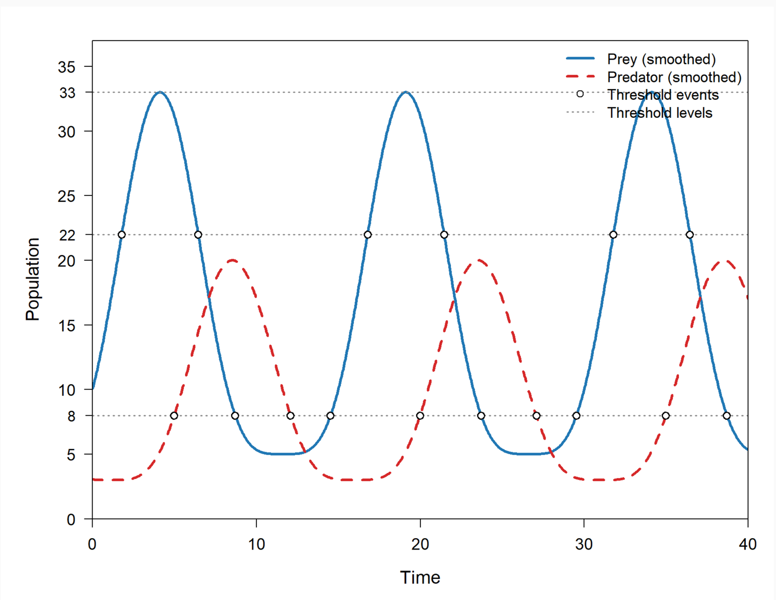
Classical ODE parameter estimation relies on repeatedly simulating complete trajectories. This is both data intensive and expensive. However, **not every observation is equally informative.**

Instead of treating the full trajectory equally, level-crossing events that carry the most useful information for parameter estimation can be identified, reducing redundant computation and focusing inference.

The framework focuses parameter estimation on the most informative parts of the trajectory, aiming to improve computational efficiency.

Level Crossings

A level crossing is an event where a population trajectory passes through a chosen population value.



Framework Pipeline

Recorded Observations

↓

Smooth the Observed Populations.
Estimate Local Derivatives

↓

Detect Threshold-Crossing Events
Interpolate state and slope at each crossing

↓

Select Informative Threshold Levels
Maximising parameter identifiability

↓

Estimate Parameters
Positive Q_n-Huber Regression on Selected Events

↓

Validate and Quantify Uncertainty
Forward Simulation and bootstrapping

↓

Parameter Estimates

Key Mathematical Results

<u>Design matrices:</u> $X_x = [x, -xy]$ and $X_y = [xy, -y]$ encode how the unknown parameters affect local prey and predator derivatives at the selected events.

$$\det(X_x^T X_x) = S_x^2 \mathrm{Var}_w(y)$$

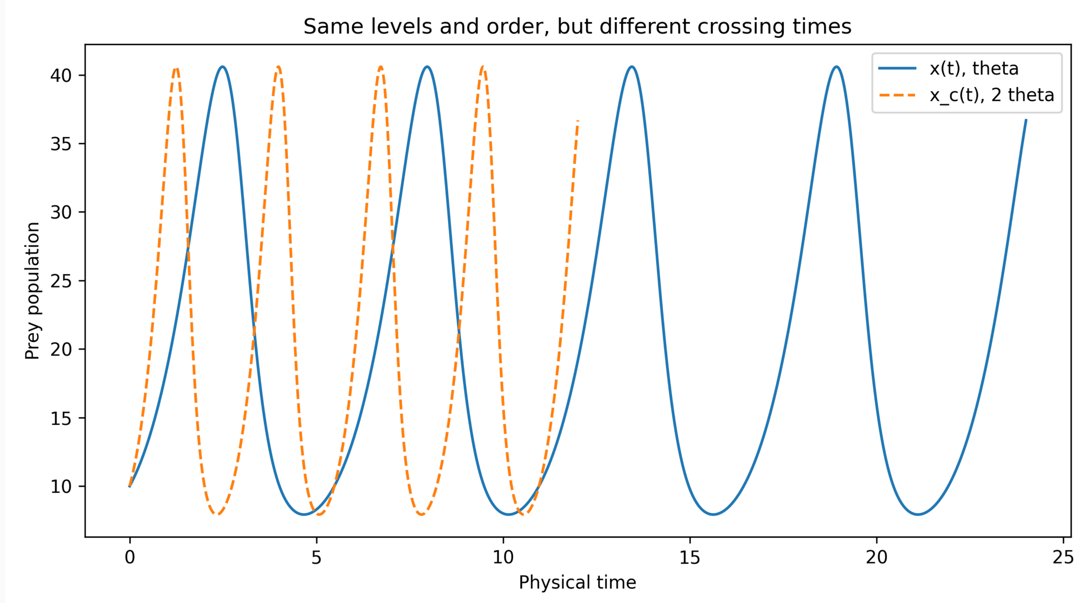
Where $S_x = \sum_{i=1}^{N} x_i^2$ and $w_i = x_i^2 / S_x$

$$\det(X_y^T X_y) = S_y^2 \mathrm{Var}_v(x)$$

Where $S_y = \sum_{i=1}^{N} y_i^2$ and $v_i = y_i^2 / S_y$

This implies that separating prey parameters needs predator variation and separating predator parameters needs prey variation.

<u>Time Scale Ambiguity:</u> Scaling all parameters by a constant preserves the state-space orbit but changes its speed: $\boldsymbol{z_{c\theta}(t) = z_\theta(ct)}$. Threshold crossing information informs the movement of the system and in what order. However, local derivative information is needed to recover the speed of the system.

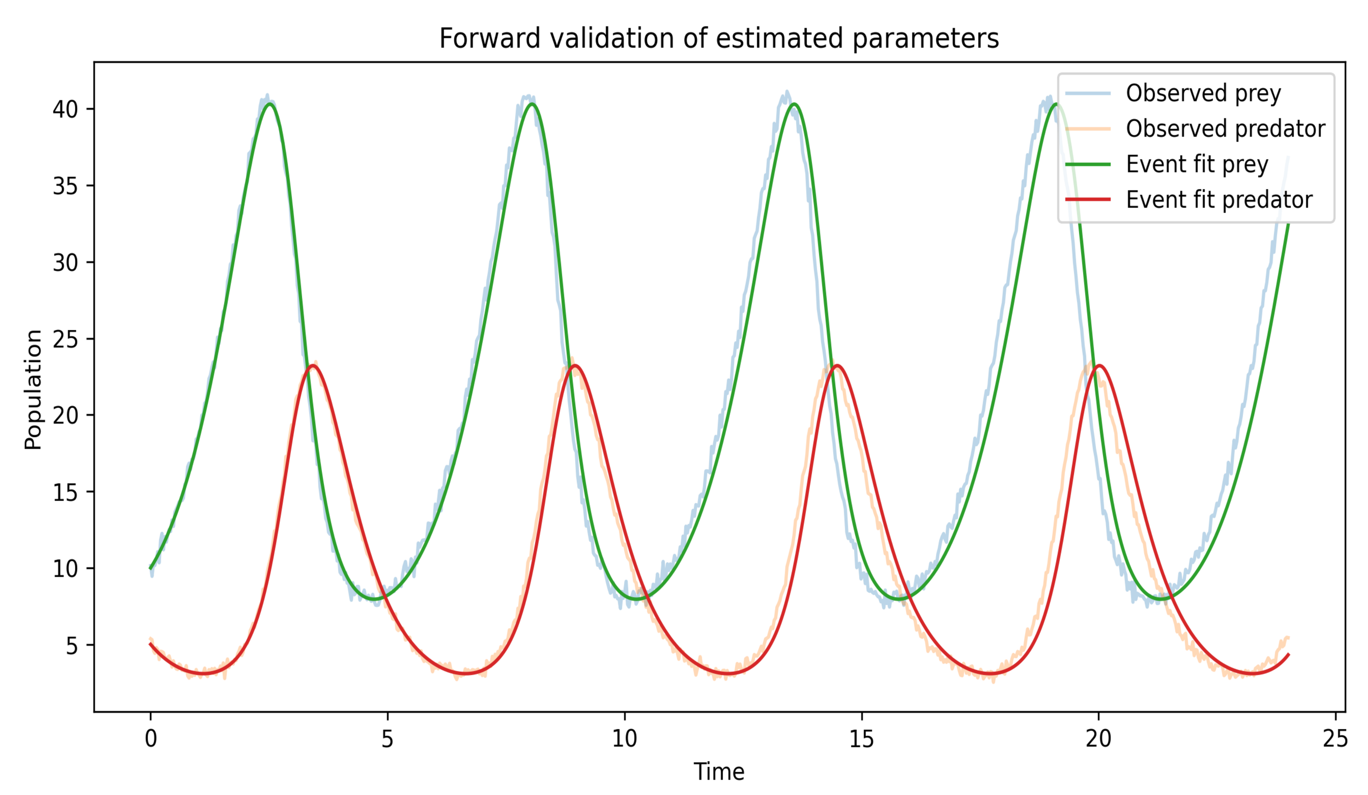


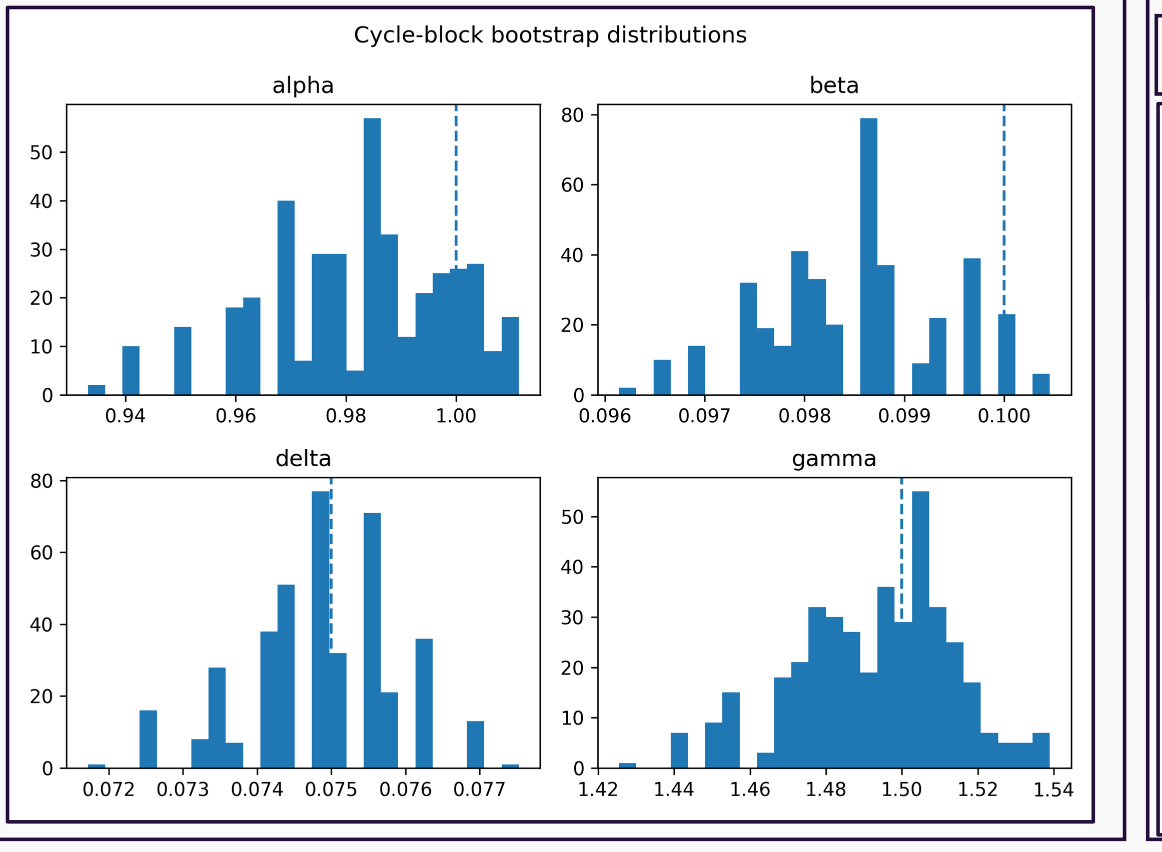
Simulation and Results

<u>Simulation:</u> A synthetic Lotka-Volterra system was generated with known parameters and initial populations (prey = 10, predator = 5). 800 observations were simulated over $t \in [0,24]$, with Gaussian noise added at 1.2% of each population's range.

<u>Key Results:</u> Parameter recovery was successful with an overall relative RMSE of 1.05%. Forward simulation reproduced the complete oscillatory dynamics with an NRMSE of 0.0566 despite being unused during fitting. Cycle-block bootstrap estimates remained concentrated near the fitted parameters over all 20 attempted and successful validation runs.

Parameter	Estimated Value	True Value	Relative Error %
α	0.9843	1.000	-1.569
β	0.09867	0.1000	-1.333
δ	0.07495	0.07500	-0.067
γ	1.494	1.500	-0.399

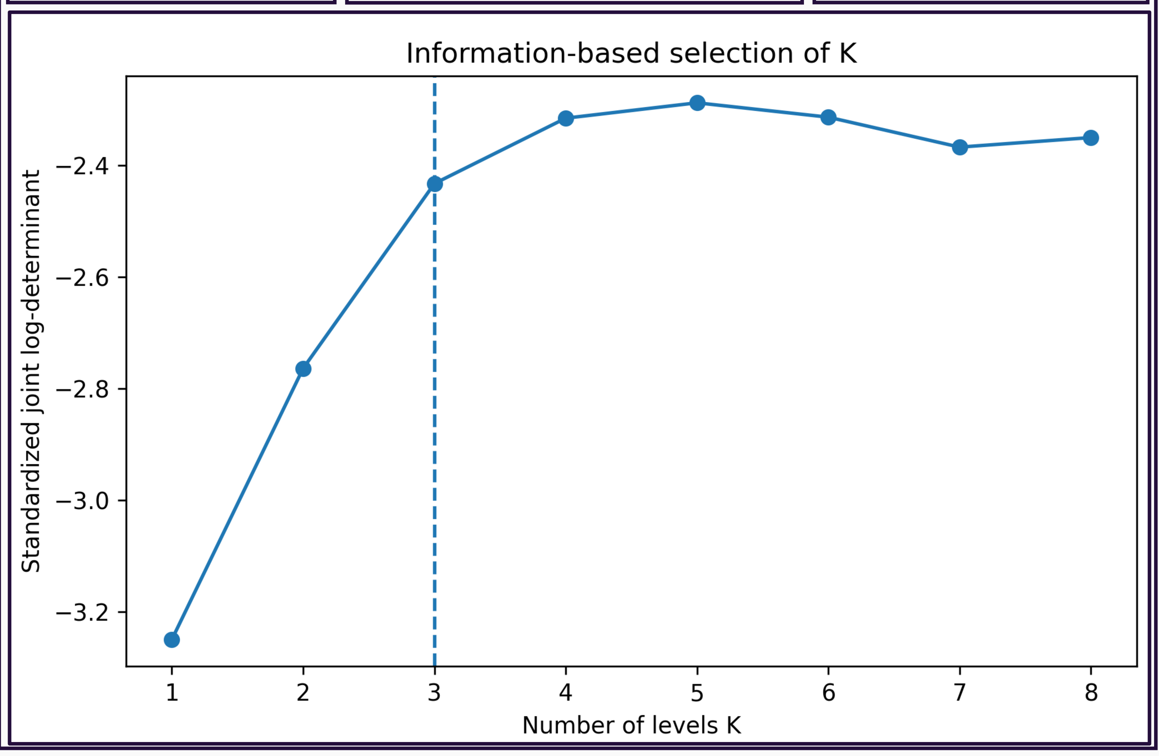




Informative Level Selection

During the simulation, **three quantile threshold levels (Q) were selected as the smallest informative set.** Adding a fourth level increased the joint score by only 4.81%, which was below the predefined 5% threshold, indicating diminishing returns.

K = 3	Q = {0.10,0.15,0.65}	50 Events



Discussion

- **Not all observations contributed equally to parameter identification.** The method selected **only 3 of 17 candidate quantile levels**, producing 50 events that retained enough cross-species variation to separate all four Lotka-Volterra parameters.
- **Accurate local fitting was sufficient to recover wider dynamics.** Parameters estimated only from selected level-crossing events reproduced the full oscillatory trajectory with a low reconstruction error.

Overall, information is governed by where the system is observed rather than how often it is observed. The selected threshold events span states that better distinguish the effects of the parameters, showing that a smaller but strategically chosen subset of observations can be more useful than a dense set of redundant points.

Key Takeaway:
Informative threshold crossings can provide a compact representation of system dynamics, allowing parameter estimation without fitting the complete trajectory.

Limitations

- Derivative estimation remains sensitive to noise and sparse sampling.
- The current validation is synthetic and should be extended to real ecological data.
- The bootstrap only captures parameter uncertainty after event selection.
- Assumes the model structure is known

Full Paper